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RANDOM GRADUALLY VARIED SURFACES FITTING*

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Let D be a digital manifold, J the subset of non-empty of D; and let $A_1, A_2, \cdots, A_m$ be m real numbers with $A_1 < A_2 < \cdots < A_m$. Supposing function $f_J: J \to A_1, \cdots, A_m$, we want to find a function $f_D: D \to \{A_1, \cdots, A_m\}$, such that f_D is gradual variation on D, having $f_D(x) = f_J(x)$, $x \in J$. Such f_D is called the gradually varied extension or interpolation of f_J.

Theorem 1. *For any two points x and y in J, if the length $d(x, y)$ of the shortest path between x and y in D is not less than the difference of gray level of the two points $LF(x, y)$ (or $d(f_J(x), f_J(y))$), then there exists a gradually varied extension f_D of f_J; and vice versa.*

Theorem 2. *If there is the gradually varied extension f_D of f_J, then there exist two gradually varied extensions f^1_D and f^2_D of f_J, such that for every gradually varied extension f_D, $f^1_D(x) \leqslant f_D(x) \leqslant f^2_D(x)$, $x \in X$.*

We gave an algorithm to compute f_D, but it made f_D in the convex closure of its guiding points. In case the holding convexity is not necessary, we should find a random gradually varied surface which can be regarded as interpolating surface between f^1_D and f^2_D.

Algorithm A (Random algorithm for f_D).

Input: D, J and f_J.

Output: gradually varied f_D or indicating no gradually varied extension for f_J.

begin

1. for (every pair p, p' in D) do
   to compute $d(p, p')$ by Floyd's algorithm;
2. for (every pair p, p' in D) do
   if ($d(p, p') < LF(p, p')$) then (no f_D and helt);
3. $D_0 := J$;
4. while ($r \in D_0$) do ($f_D(r) := f_J(r)$);
5. using depth-first-searching algorithm, to connect all
   points in D except D_0, denoted by H;

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6. if ($D_0 = D$) then (output f_D and helt) else
begin
choose x that is the endpoint of H and choose $r \in D_0$, $d(r, x) = 1$,
without loss of generality assume that $f_D(r) = A_i$,
$v(1) := A_{i-1}$, $v(2) := A_i$, $v(3) := A_{i+1}$;
for $t := 1$ to 3 do
7. begin
while ($p \in D_0$) do if ($d(x, p) < d(v(t), f_D(p))$) goto 8,
$v(t) := 1$;
8. end
$R := \{t \mid v(t) = 1\}$
random choose $j \in R$
$j := i + j - 2$, $f_D(x) := A_j$
$H := H - \{x\}$
$D_0 := D_0 \bigcup \{x\}$
goto 6
end
end

Lemma 1. *The computing time of Algorithm* A *is* $O(|D|^3)$.

There is a defect in Algorithm A, i. e. its valuation procedure has an order which was determined beforehand; consequently the randomness of the surface is restricted. The following algorithm A′ such that $x \in D$ is also random.

Algorithm A′

Input, Output and Step 1 until Step 4 are similar to those in Algorithm A.
begin
....
5. random choose x in $D - D_0$;
6. for $t := 1$ to m do begin
while ($p \in D_0$) do if ($d(x, p) < d(A_t, f_D(p))$) goto 8
$v(t) := 1$;
8. end
$R := \{t \mid v(t) = 1\}$
random choose $j \in R$
$f_D(x) := A_j$
$D_0 := D_0 \bigcup \{x\}$
goto 5
end

Theorem 3. *f_D is gradually varied which is obtained by Algorithm* A′.

Proof. We need only to prove that if there is a gradually varied extension for f_J, then output f_D of Algorithm A′ is gradual variation. Suppose there is a gradually varied extension. Therefore $\forall x, y \in J$, $d(x, y) \geqslant LF(x, y)$. In the following, we shall prove that an integer t exists such that $v(t) = 1$ at Step 6 of Algorithm A′.

On the contrary, let such t do not exist, i. e.

$$\forall\ t \in \{1, \cdots, m\},\ \ \exists\ y \in D_0,\ d(x, y) < d(A_t, f_D(y)). \tag{1}$$

However, according to Theorem 1, if

$$\forall\ p, q \in D_0,\ d(p, q) \geqslant d(f(p),\ f(q)), \tag{2}$$

then there exists an extension $f: D \to \{A_1, \cdots, A_m\}$, such that

$$\forall\ p, q \in D,\ d(p, q) \geqslant d(f(p),\ f(q)). \tag{3}$$

which is in contradiction with (1). Thus, such t exists, and it is easy to see that f_D is gradual variation on D.

Only $|D|$ elements in $A_1, \cdots, A_m$ are efficient if $m > |D|$. Therefore, we have

Lemma 2. *The computing time of Algorithm* A′ *is* $O(|D|^3)$.

Let Σ_n be an n-dimensional grid space, and D the connected subset of Σ_n. If D is the convex set, then it is not necessary that the length of the shortest path is computed by Floyd's algorithm, and Algorithms A and A′ are simplified relevantly.

Theorem 4. *Let D be the convex set in Σ_n. Algorithm* A *needs* $O(|D|^2)$ *time, but Algorithm* A′ *requires still* $O(|D|^3)$ *time.*

If Σ_2 is 4-adjacent, i. e. $D \subseteq \Sigma_2$ has Jordan property, then much faster algorithm can be designed.

Corollary 1. *Let D be the Jordan square in Σ_2. Then there exists an* $O((|D| - |J|)\log_2(|D| - |J|) + |J|\sqrt{|D| - |J|}\ \log_2(|D| - |J|))$ *time improved Algorithm* A.

Given the distribution of f_D in $\{A_1, \cdots, A_m\}$, let $\xi(f_D, A_i) = \xi_i$, $\sum_{i=1}^{m} \xi_i \neq |D|$ and $P(A_i) = \xi_i / |D|$. For convenience, we say $\{\xi_i\}$ is the distribution. Set $N(S) = \{p: p \in D - S$ and p is adjacent to some points of $S\}$, and then $N(S)$ is called the adjacent point set of S.

Lemma 3. *For any S with $|S| = \{\xi_i\}$, if $N(S) > \xi_{i-1} + \xi_{i+1}$, then there are no gradually varied surfaces with distribution $\{\xi_i\}$.*

Not considering gradual variation, there are valuations of $|D|! / \prod_{i=1}^{m} \xi_i!$ kinds on D with distribution $\{\xi_i\}$. That is to say, it is impossible to construct gradually varied surfaces by one-by-one testing method.

Theorem 5. *There is no algorithm with polynomial-time to construct a gradually varied* *surface whose distribution was given beforehand unless* $P=NP$.

Proof. We need only to transform an NP-complete problem to our problem in polynomial time[2]. PARTITION problem is selected: Let $R_1, \cdots, R_n$ be n sets, and $N=\{1, \cdots, n\}$. Is there a subset $N' \subset N$ such that $\sum_{i \in N'} |R_i| = \sum_{i \in N-N'} |R_i|$?

For any $i \in N$, connect any two points in R_i, where R_i and R_j are not adjacent, $i \neq j$. Suppose a point $r \notin \bigcup_{i=1}^{n} R_i$, such that r is connected with every point in $\bigcup_{i=1}^{n} R_i$. Thus, we obtain a special digital manifold. Let $\{A_i\}=\{A_1, A_2, A_3\}$ and $\{\xi_i\}=\left\{\sum_{i=1}^{n}|R_i|/2, 1, \sum_{i=1}^{n}|R_i|/2\right\}$. Obviously, the valuation of r is only A_2, or else f_D is not gradually varied or does not satisfy distribution $\{\xi_i\}$. The above procedure can be run in polynomial time.

Thus, there exists a gradually varied surface with distribution $\{\xi_i\}$ on $D=\{r\} \cup \bigcup_{i=1}^{n} R_i$ if and only if there is $N' \subseteq N$ such that $\sum_{i \in N'} |R_i| = \sum_{i \in N-N'} |R_i|$. That is to say, we turn the PARTITION problem to construction (or decision) problem of a gradually varied surface whose distribution was given by polynomial transformations.

According to Theorem 5, we only obtain the gradually varied surface approximating the distribution. However, in some simple case, it is possible to construct a gradually varied surface with distribution $\{\xi_i\}$ such as

Lemma 4. *For a rectangle grid* D *in plane, if* $\xi_1 < \cdots < \xi_m$ *or* $\xi_1 > \xi_2 > \cdots \xi_m$, *there exists a gradually varied surface with distribution* $\{\xi_i\}$.

Let D_0 be the simply connected subset of D and $f(D_0)$ be single valued, such as A_i. If $N(D_0)$ and the boundary ∂D of D do not intersect, D_0 is called the extreme region.

Lemma 5. *For a Jordan digital manifold* D *in which the number of adjacent points of every point is nearly identical, if* S *is a connected subset with* $|S|=n$ *such that* $|N(S)|$ *is minimal, then for any* $S_1, S_2 \subseteq D$ *and* $|S_1|+|S_2|=n$, *it holds that*

$$|N(S)| \leqslant |N(S_1)| + |N(S_2)|.$$

By Lemma 5, we approach an algorithm for the gradually varied surface with one extreme region whose distribution was given as follows.

Algorithm B (The approached algorithm for the gradually varied surface with one extreme region).

Input: Jordan set D, $\{A_i\}$ and $\{\xi_i\}$; we may suppose $\xi_1 \leqslant \xi_m$.

Output: surface f satisfying distribution $\{\xi_i\}$.

```
begin
  S: =∅
  for i: =1 to m do begin
    for j: =1 to ξ(i) do begin
      choose x, S: =S ⋃ {x} such that |N(S)|=min
      f(x): =A_i
    end
  end
end.
```

Algorithm B is in $O(|D|^3)$ time, but f may be not gradually varied unless D is a closed manifold. The following technique can be used for any surface f, to compute the gradually varied surface g which approximates to f uniformly.

Let $F_i(p)=\{f(p)+t|t=0, \pm 1, \cdots, \pm i\}$, here might as well let $A_i=i, p\in D$; and let

$$F_i(p)/F_i(q)=\{x|x\in F_i(p),\ \text{there is } y\in F_i(q),\ \text{such that } d(p, q)\geqslant d(x, y)\}, \quad (4)$$

and

$$\begin{cases} F_i^{(k)}(p)=\bigcap_{q\in J} F_i^{(k-1)}(p)/F_i^{(k-1)}(q), \\ F_i^{(0)}(p)=F_i(p). \end{cases} \quad (5)$$

We can prove that if

$$\exists\, k, \forall p\in D, F_i^{(k)}(p)=F_i^{(k+1)}(p)\neq\emptyset. \quad (6)$$

then for every $p\in D$, setting $g(p)=\inf\{F_i^{(k)}(p)\}$, we know that g is gradual variation and $\forall p\in D$, $d(f(p), g(p))\leqslant i$. Consequently, let $i=1, 2, \cdots$ until (6) is satisfied. Then we can obtain the optimal uniformly approximation g of f. This method has been used for stratum surface fitting[3].

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