# THE NECESSARY AND SUFFICIENT CONDITION AND THE EFFICIENT ALGORITHMS FOR GRADUALLY VARIED FILL*

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Filling is an important part in pattern recognition and computer vision. In the discrete plane (grid of plane) $\Sigma_2$, fill means "determination of the region $D$ enclosed by the simple closed curve $J$ which is given beforehand"[1,2]. In other words, if the valuation of the point on the contour $J$ is 1, we want to do a valuation for $\Sigma_2$ to make the valuation of the point $p$ be 1 iff point $p$ belongs to $D$.

But in reality, values of the contour $J$ are variable, it is not single valued. In most cases, values of the contour $J$ are varying gradually. That is to say, not only the region $D$ should be determined but also the value of each point in $D$ should be determined and such values should be proved to be gradually varied. In the present note, only the second part will be examined. As for the first part, it may be seen in Refs. [1] and [2].

In continuous spaces, this problem is trivial. If $J$ is a Jordan curve in plane and $f: J \to R$ is the continuous function (or belongs to $C^{(n)}$), obviously there is the continuous ($C^{(n)}$) extension of $f$ on $D$. Whereas, the result is not true in a discrete space. On the contrary, the concerned results in continuous space are corollaries in discrete space.

Note that, in a discrete plane, some of the simple closed curves are not Jordan curves, it is dependent on the definition of connectivity (4-adjacent is Jordan, 8-adjacent is not)[3,4]. We will see that it is very important for the design of the algorithm for gradually varied fill and its time complexity.

Given the formal definition of gradual variation. Let $A_1, A_2, \cdots, A_m$ be the $m$ rational numbers (or real numbers), and $A_1 < A_2 < \cdots < A_m$, i.e. $\{A_1, A_2, \cdots, A_m\}$ gives the system of gray level. And let $f: \Sigma_2 \to \{A_1, \cdots, A_m\}$ be the gray scale.

We say that $f$ is gradually varied on two adjacent points $p$ and $p'$ of $\Sigma_2$ (or other discrete space), if $f(p) = A_i$ then $f(p')$ is $A_{i-1}$, $A_i$ or $A_{i+1}$. $f$ is gradually varied on digital

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manifold if $f$ is gradually varying on any two adjacent points of digital manifold (curve, surface, etc. ).

Repeat two basic concepts as follows:

(1) Length of path. Let $P=p_1, p_2, \cdots, p_n$ be the path, the length of $P$ is $n-1$. The length of the shortest path between $p$ and $p'$ is expressed by $d(p,p')$.

(2) Difference of gray level. Suppose $f(p)=A_i$ and $f(p')=A_j$, the difference of gray level on $p$ and $p'$ is $|i-j|$. It is denoted by $LF(p,p')$.

Let $D$ be the simply connected set in the discrete space (such as $\Sigma_2$) and the contour $J$ of $D$ be simply closed. If $f_J: J \to \{A_1, \cdots, A_m\}$ is gradually varied on $J$, the problem of gradually varied fill is defined as: there is an extension of $f_S$ on $D$, $f_D: D \to \{A_1, \cdots, A_m\}$, such that $f_D$ is gradually varied on $D$. Here the extension means $f_D(p)=f_J(p)$ if $p \in J$.

**Theorem.** *The necessary and sufficient condition under which there exists the gradually varied fill on D is that for any two points p and p' in J, the length of the shortest path between p and p' in D is not less than the difference of gray level of p and p'.*

*Proof.* (1) Suppose $f_D$ is the gradually varied function on $D$, then $f_D$ is gradually varied on every curve (path) in $D$. The curve may be the shortest path between two points $p$ and $p'$ in the contour $J$ of $D$. Hence, the length of the shortest path is not less than the difference of gray level of $p$ and $p'$, i.e. $d(p,p') \geqslant LF(p,p')$. (2) Conversely, suppose $d(p,p')$, if $p, p' \in J$ in $D$, the following proof is consturctive (an algorithm). For every $p \in J$, let $f_D(p)=f_J(p)$ and $f_D(p)=\theta$ and if $p \in D-J$, denote

$$D_0=\{p: f_D(p) \neq \theta,\ p \in D\}.$$

For any $p$, $p' \in D_0$, we have $d(p,p') \geqslant LF(p,p')$.

(a) If $D_0 \neq D$, we can find a point $r \in D_0$, such that there is an adjacent point $x$ of $r$, $x \in D-D_0$. We may as well suppose $f_D(r)=A_i$.

(b) Let $f_D(x)=f_D(r)=A_i$, and let

$$m(x)=\{p: f_D(p)<f_D(x),\ p \in D_0\}.$$

$$M(x)=\{q: f_D(q)>f_D(x),\ q \in D_0\}.$$

There exist three cases:

(i) If there is a $p \in m(x)$, such that $d(x,p)<LF(x,p)$. $d(r,p) \leqslant d(r,x)+d(x,p)$ and $d(r,x)=1$, hence $d(r,p) \leqslant 1+d(x,p)$ and $d(x,p) \geqslant d(r,p)-1$. In addition, $d(r,p) \geqslant LF(r,p)$ and $LF(r,p)=LF(x,p)$, thus $d(x,p) \geqslant d(r,p)-1 \geqslant LF(x,p)-1$. According to the assumption $d(x,p)<LF(x,p)$, hence $d(x,p)=LF(x,p)-1$. For any $q \in M(x)$, $f_D(p)<f_D(x)<f_D(q)$, so $LF(p,q)=LF(p,x)+LF(x,q)$. Again $d(p,x)+d(x,q) \geqslant d(p,q) \geqslant LF(p,q)$, then

$$d(p,x)+d(x,q)\geqslant LF(p,x)+LF(x,q)\geqslant d(p,x)+1+LF(x,q),$$

hence $d(x,q)\geqslant LF(x,q)+1$. Thus, we have proved that if there exists $p\in m(x)$ such that $d(x,p)<LF(x,p)$, then we have

$$\forall p\in m(x)\ (d(p,x)\geqslant LF(p,x)-1)\ \&\ \forall q\in M(x)\ \ (d(x,q)\geqslant LF(x,q)+1).$$

We modify the valuation of $x$, let $f_D(x)=A_{i-1}$. Obviously, for the new value, we have

$$d(y,x)\geqslant LF(y,x),\ \text{if}\ \ y\in D_0\ \ \&\ \ f_D(y)\neq A_i.$$

$$d(y,x)\geqslant LF(y,x)=1,\ \text{if}\ \ y\in D_0\ \ \&\ \ f_D(y)=A_i;$$

(ii) If there is $q\in M(x)$, such that $d(x,q)<LF(x,q)$. Similarly, we have $d(x,q)\geqslant LF(x,q)-1$ for any $q\in M(x)$, and $d(p,x)\geqslant LF(p,x)+1$ for any $p\in M(x)$. Hence, modifying $f_D(x)=A_{i+1}$, we obtain $\forall y\in D_0\,(d(x,y)\geqslant LF(x,y))$.

(iii) If (i) and (ii) are not satisfied, then $f_D(x)=A_i$ will just be required.

(c) Let $D_0\leftarrow D_0\bigcup\{x\}$, thus for any $p$ and $p'$ in $D_0$, we have

$$d(p,p')\geqslant LF(p,p').$$

Return to (a), and loop this process until $D_0=D$. For any $x,y$ in $D$, if $x$ and $y$ are adjacent, i. e. $d(x,y)=1$ because $d(x,y)\geqslant LF(x,y)$, then $LF(x,y)=0$ or 1. That is, $f_D$ is gradually varied on $x$ and $y$, then $f_D$ be gradually varied on $D$.

In the above proof, we do not make particular restriction except using the symmetric relation and the triangle inequality of distance. Hence, this theorem may be generalized to finite digital manifold. The digital manifold means the "discretization or net sampling" of some manifold.

**Corollary 1.** *Let $D$ be the simply connected digital manifold, and the boundary $J$ of $D$ be simply closed. Suppose $f_J: J\to\{A_1,\cdots,A_m\}$ is gradually varied on $J$. Then there is a gradually varied extension of $f_J$ on $D$, such that $f_D: D\to\{A_1,\cdots A_m\}$ iff the length of the shortest path between any two points $p$ and $p'$ in $J$ is not less than the difference of gray level of $p$ and $p'$ in $D$.*

Now we discuss the algorithms for gradually varied fill which can be distinguished into two kinds, the first kind is the decision algorithm by which we solve whether "there is the gradually varied fill". The other is the fill itself, or we call the constructive algorithm. In fact, we have already shown these two kinds of algorithms in the proof of the theorem. According to Dijkstra's algorithm and Floyd's algorithm for finding the shortest path, it is not difficult to prove Corollary 2.

**Corollary 2.** (1) *There is a decision algorithm which has the time complexity $O(|J||D|^2)$.*

(2) *There is the $O(|D|^3)$ time algorithm for gradually varied fill.*

Under usual instance, the number of adjacent points of a point is finite in discrete spaces; and the distance of two points may be calculated in a constant (or linear) time. Thus, we can introduce many concrete results (see "On Gradually Varied Fill Algorithms"). Particularly, if $D$ has Jordan property, i.e. a simple closed curve or super-surface segments the rest of $D$ into two disconnected parts (or several components in discrete case). The illustration is shown in Fig. (1). Here, $\forall x \in D_1 - \{J \bigcup J_1\}$ & $\forall y \in D_2 - \{J \bigcup J_1\}$, every path to join $x$ and $y$ must contain the point of $J_1 \bigcup J$. Thus, if $f_1$ is gradually varied on $D_1$ and $f_2$ is on $D_2$, for any $x \in J$, we have $f_1(x) = f_2(x)$. Then $f$ is gradually varied on $D$ that is composed by $f_1$ and $f_2$. This method is similar to the divide-and-conquer techniques, i.e. we first define the valuation of the shortest path between $p$ and $p'$, then consider $D_1$ and $D_2$. If $D$ is defined by 8-adjacent way (non-Jordan see Fig. (2)), then $f$ is not gradually varied on $D$, here $f$ is composed of $f_1$ and $f_2$. Jordan property has essential sense for us to design fill algorithms.

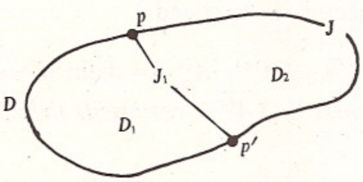


Fig. 1

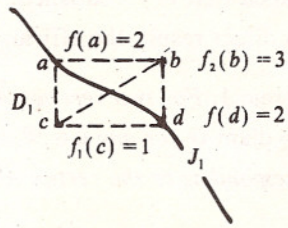


Fig. 2

**Corollary 3.** *Let D be a square in 4-adjacent $\Sigma_2$, then there is an $O(|D| \cdot \log_2 |D|)$ time algorithm for gradually varied fill.*

Actually, these results may be generalized in the sense of $\lambda$-connection (or continuity), and can be used for intelligent data fitting[5,6]. The gradually varied fill is also a technique of surface construction (interpolation) under which boundary valuations are known. Like Bezier polynonmial fitting and *B*-spline, the gradually varied function (surface) which is constructed by the algorithm in this note has a good property.

**Corollary 4.** *The gradually varied surface is in the convex closure of its guiding points.*

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