A Fast Algorithm for Homology Groups in 3D Cubical Space

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Abstract

This talk presents a linear time algorithm to compute homology groups of 2D or 3D objects in 3D cubical space. Without pre-calculating triangulation of real data, our algorithm will directly use the geometrical and topological properties of digital space to get local surface points, calculate the the genus, and homology groups. This algorithm is in linear time. At the last, we will show some data examples

Introduction

- Digital Space is important to Computer Science.
- Definition of Digital Spaces
- Graphs
- Connectivity in Digital Space
- Digital Curves in 2D
- Digital Surfaces in 3D

Why digital spaces

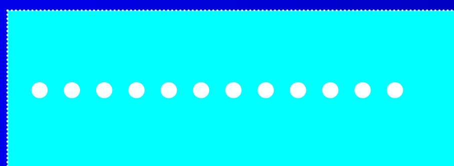
- Computer only can store and process things in digital /discrete forms.
- Computer memory is a digital array.
- Digital cameras capture digital images that are stored in 2D digital space.
- Image intensity are quantified as digital level.
- Video are sequential images

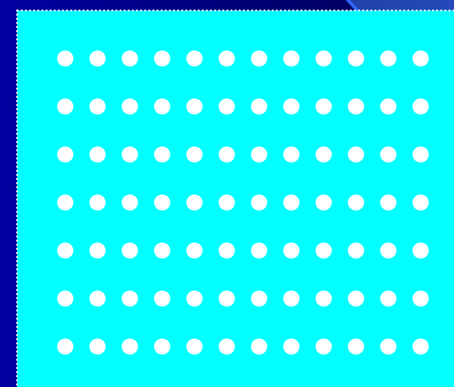
Definition of Digital Space

- Digital space usually means the grid space or formed by integer vectors.
- Digital space contains a measure of common units:
- Unit length of digital space is fixed.
- Discrete space are form by discrete objects in Euclidean space.
- Discrete space does not have common units

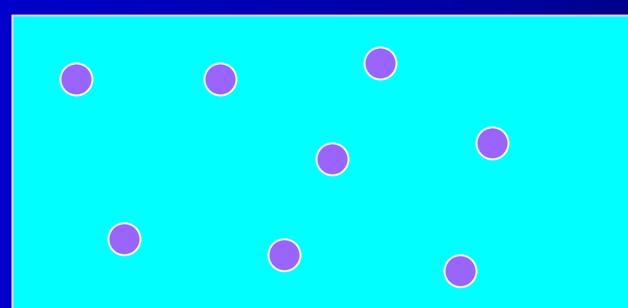
Examples of Digital Space and Discrete Space

- 1D digital space and 2D digital space



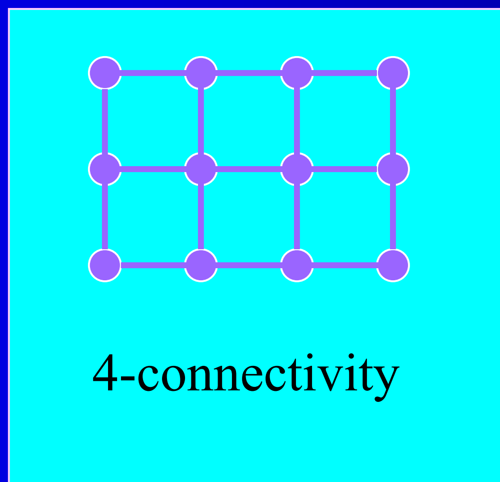


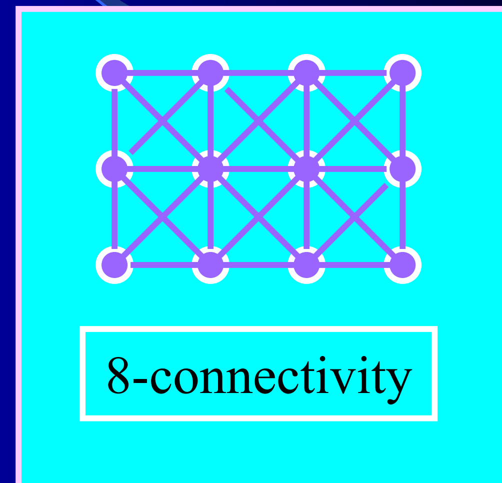
- Discrete space



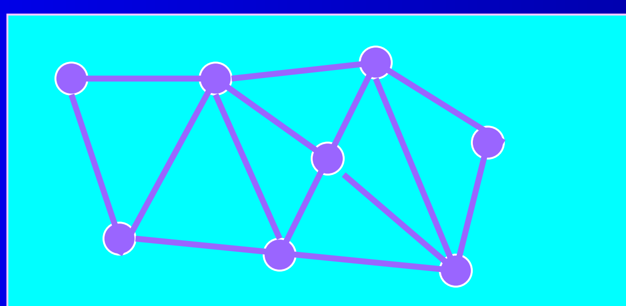
Connectivity

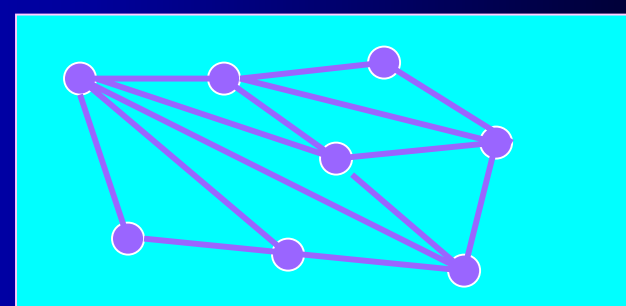
- Digital space has regular connectivities





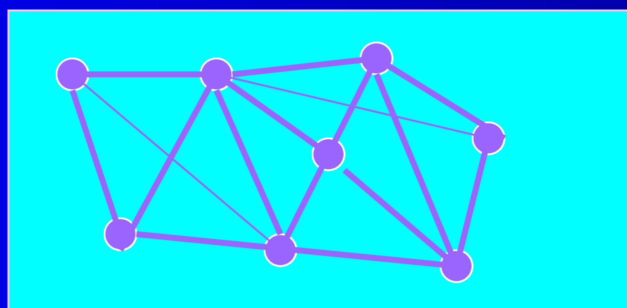
- Discrete space has multiple interpretations

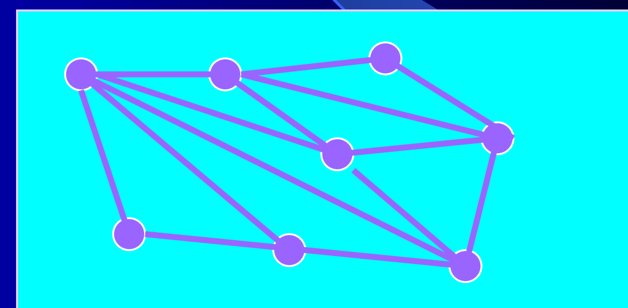




General Graphs

- **The Digital space and the discrete space are special graphs**





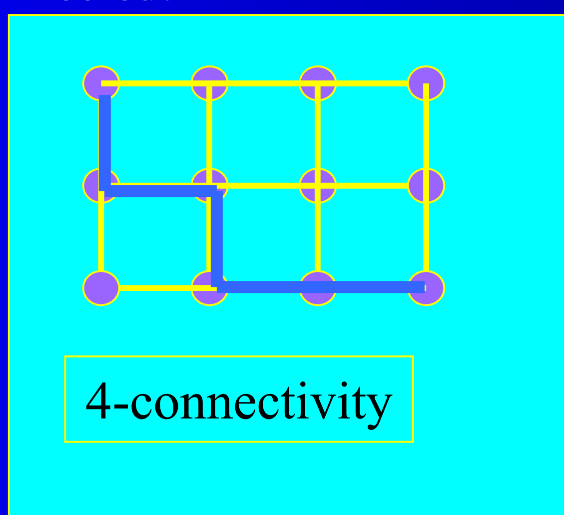
- **What is a graph? Refer to discrete math.**

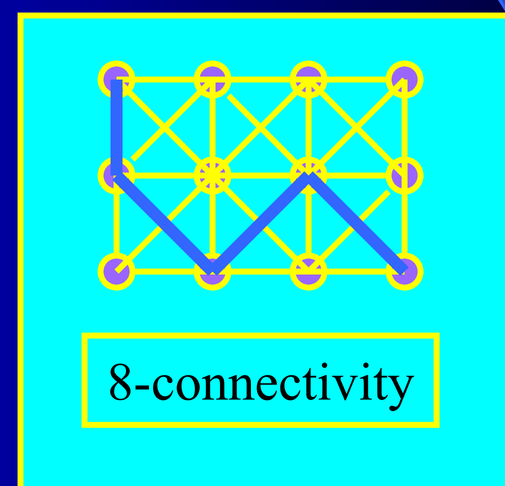
Digital Lines and Digital Curves

- **Perfect Digital lines are only few**
- **Bresenham's digital line are approximation of continuous lines in digital space using Bresenham's line algorithm in computer graphics**
- **Digital curves are paths of grid (digital) points.**

2D Digital Space

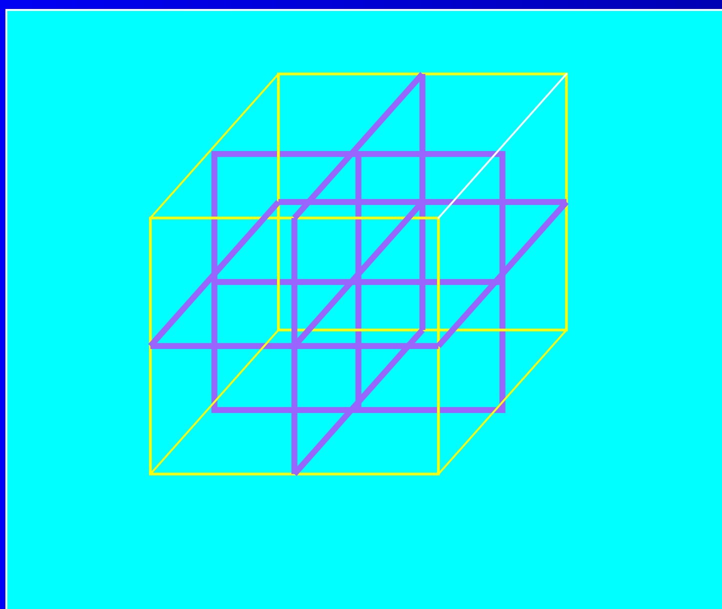
- Two points are connected if there is a path that connects them
- A closed curve is a closed path
- Connectivity of the path can be 4-connected or 8-connected.
- Connectivity are the equivalence relation which satisfies: reflexivity, symmetry, and transitivity.
- A connected component is the set where every pair of points are connected.

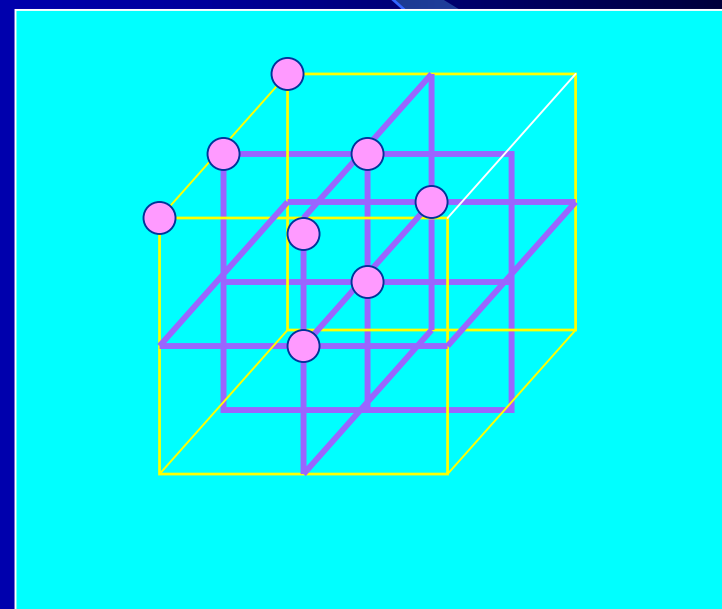




3D Digital Space with 6-connectivity

- Three dimensional space





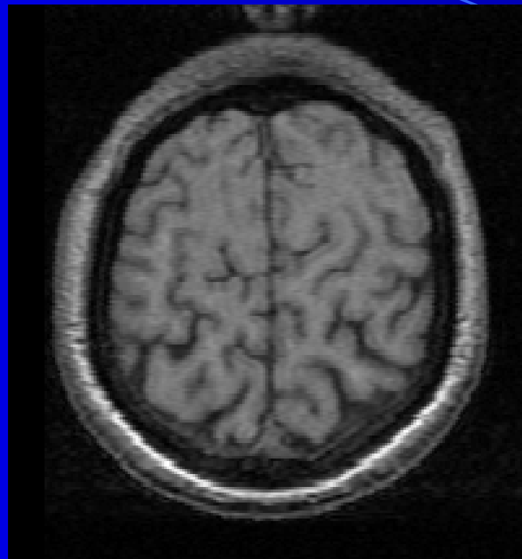
Why Digital/discrete Surfaces (1)

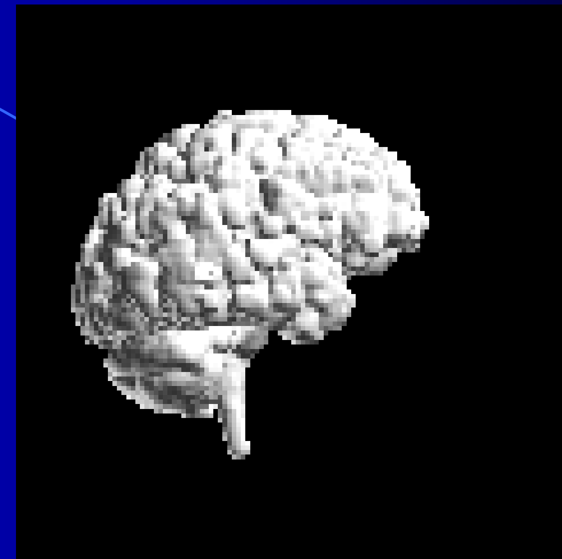
- Edge detection and 3D object tracking.

  Medical image examples:

  MR Brain images

- The result of the extraction will be digital surface.





Example:

MR Slice Image and segmented or tracked 3D brain

(Grégoire Malandain)

Why Digital/discrete Surfaces (2)

- A 2D gray scale image is a digital surface with or without continuity.

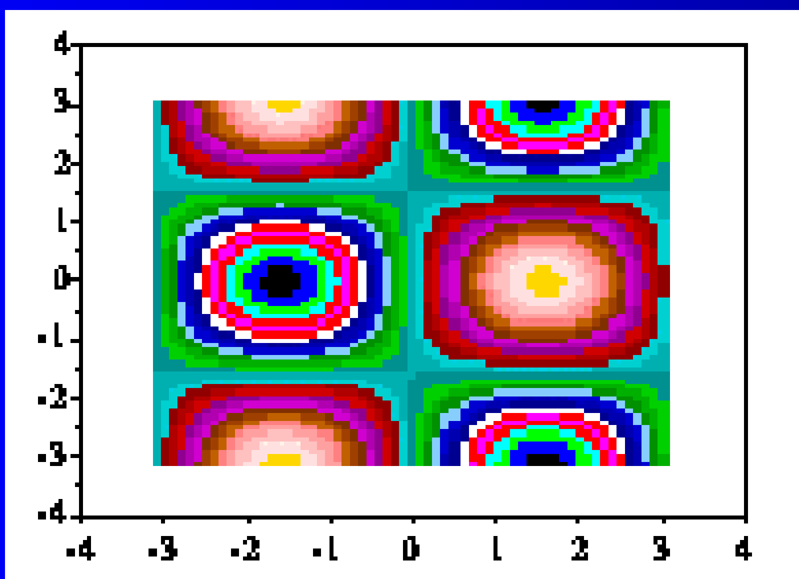
  Examples

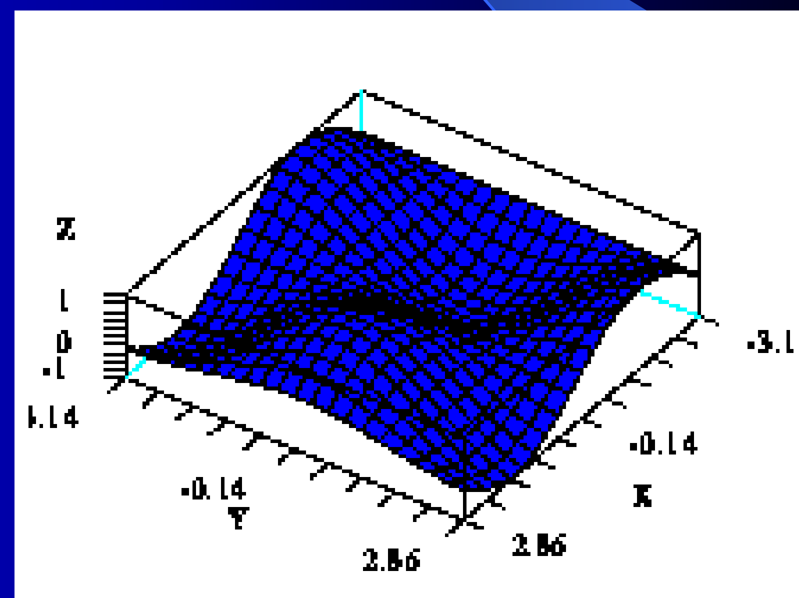
- An object in the image is often appeared to be a “continuously looking” segment.

  Or an object would be a “continuous” digital surface on a segmented region.

Example:

Gray Scale Image vs. Digital Surface





The Digital Surface and Manifold(1)

Definition of 3D digital surfaces:

- Artzy, Frieder, and Herman: *A digital surface is the boundary of a 3D digital object*. (Intuitive)

The Digital Surface and Manifold (2)

- Morgenthaler and Rosenfeld: *A digital surface is the set of surface points each of which has two adjacent components not in the surface in its neighborhood.* (Set-theoretic)
- Chen and Zhang: *A digital surface is formed by moving of a line-segment.* (Dynamic & recursive)

The Digital Surface and Manifold (3)

Basic Concepts:

- A point is 0-cell, a line segment is 1-cell, etc.
- An (i+1)-cell can be formed by two disjoint i-cells that are parallel. Or,
- An i-cell and its parallel move form an (i+1)-cell.

The Digital Surface and Manifold (4)

Definition of digital manifolds (Chen and Zhang, 1993): A connected subset S in digital space Σ is an i_D digital manifold if (give example for i=2):

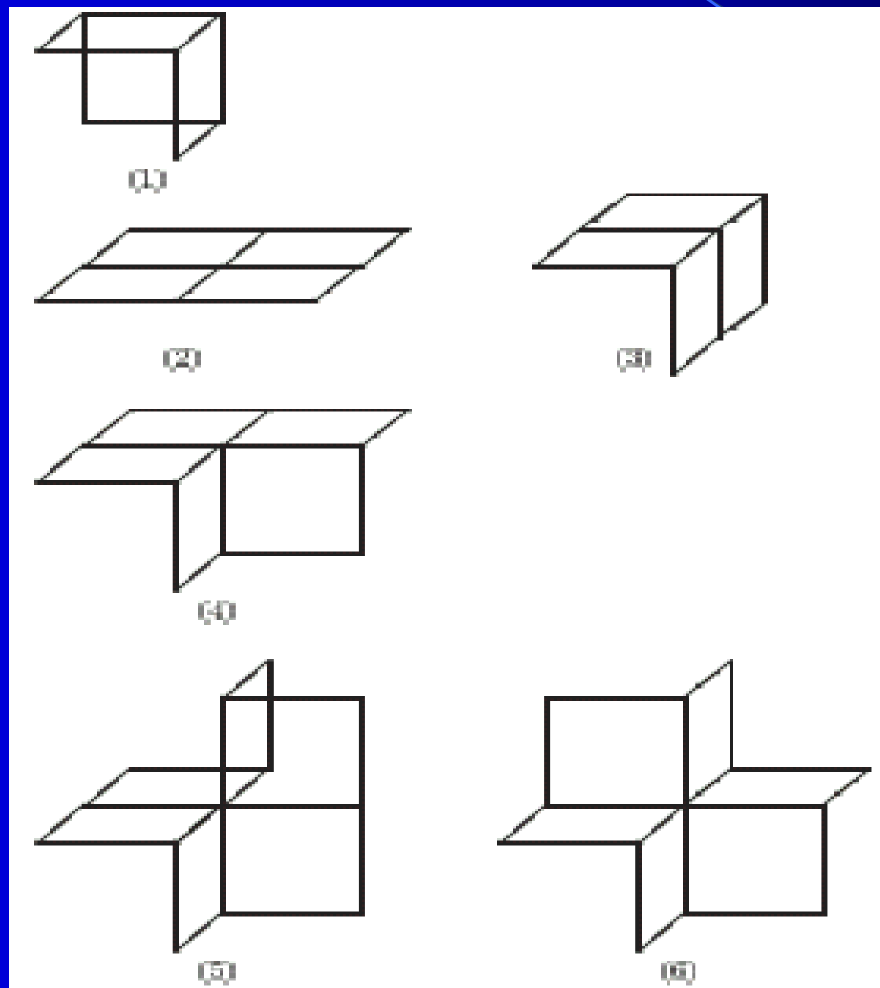
1) *Any two i-cells are (i-1)-connected in S,*
2) *Every (i-1)-cell in S has only one or two parallel-moves in S, and.*
3) *S does not contain any (i+1)-cell.*

Classification: Digital Surface Points in 3D

Chen *et al* obtained:

- **Theorem**: *The Morgenthaler-Rosenfeld's surface is equivalent to the surface defined by Chen and Zhang (in direct adjacency).*
- **Theorem:** *There are exactly 6 types of digital surface points in 3D (in direct adjacency).*

Classification(continue):
6 types of digital surface Points



Gauss-Bonnet Theorem and Closed Digital Surfaces

- Assume that Mi (M3, M4, M5, M6) is the set of digital points with i neighbors.
- The Gauss-Bonnet theorem states that if M is a closed manifold, then

$$\int_M K_G dA = 2\pi\chi(M)$$

$$\Sigma_{\{p \text{ is a point in } M\}} K(p) = 2\pi \cdot (2 - 2g)$$

Gauss-Bonnet Theorem and Closed Digital Surfaces (Continue)

According to discrete Gaussian Curvature Theorem . The curvature K of the center point of the polyhedra is determined by

$$2\pi - \Sigma_i \theta_i$$

K3 = Pi/2; K4=0 ; K5 =-Pi/2; K6 = −Pi.

Therefore

$$g = 1 + (|M_5| + 2 \cdot |M_6| - |M_3|)/8.$$

Homology Groups of 3D Objects

Theorem 3.4 *Let M be a compact connected 3-manifold in S^3. Then*

(a) $H_0(M) \cong Z$.

(b) $H_1(M) \cong Z^{\frac{1}{2}b_1(\partial M)}$, *i.e.* $H_1(M)$ *is torsion-free with rank being half of rank* $H_1(\partial M)$.

(c) $H_2(M) \cong Z^{n-1}$ *where* n *is the number of components of* ∂M.

(d) $H_3(M) = 0$ *unless* $M = S^3$.

The Key is to calculate the Betti Number b1 that is the summation of the genus in all connected components in boundary of M.

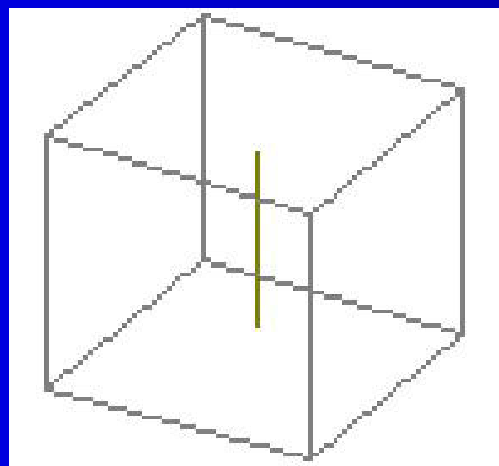
Linear Algorithm for Homology Groups of 3D Objects in 3D

Step 1. Track the boundary of M, ∂M, which is a union of several closed surfaces. This algorithm only needs to scan though all the points in M to see if the point is linked to a point outside of M. That point will be on boundary.

Step 2. Calculate the genus of each closed surface in ∂M using the method described in Section 2. We just need to count the number of neighbors on a surface. and put them in M_i, using the formula (5) to obtain g.

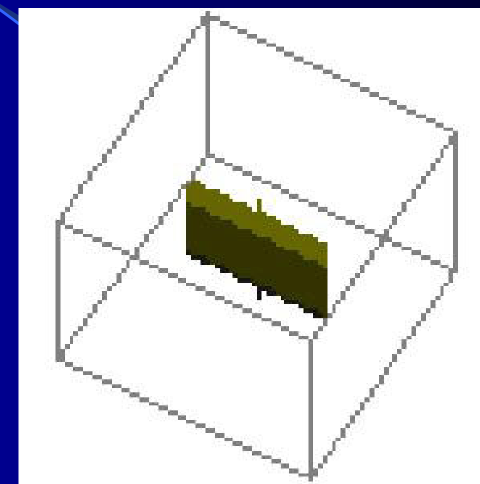
Step 3. Using the Theorem 3.4, we can get H_0, H_1, H_2, and H_3. H_0 is Z. For H_1, we need to get $b_1(\partial M)$ that is just the summation of the genus in all connected components in ∂M. (See [7] and [6].) H_2 is the number of components in ∂M. H_3 is trivial.

Focus of the Genus for boundary of 3D Objects (Raster Space)



M3= 8 ; M4=172 ; M5=0; M6= 0

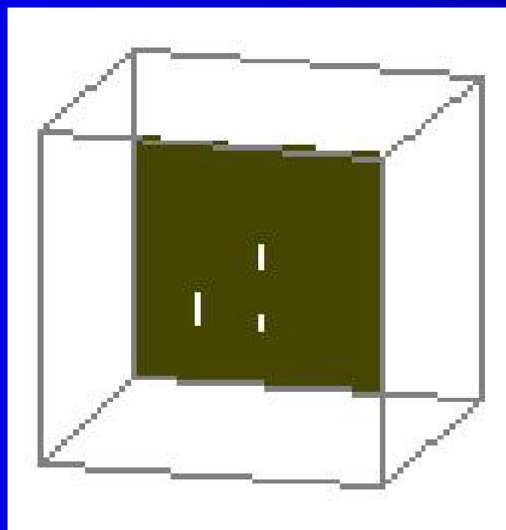
Boundary Genus= 0



M3= 16 ;M4=2466; M5= 8 ; M6=0

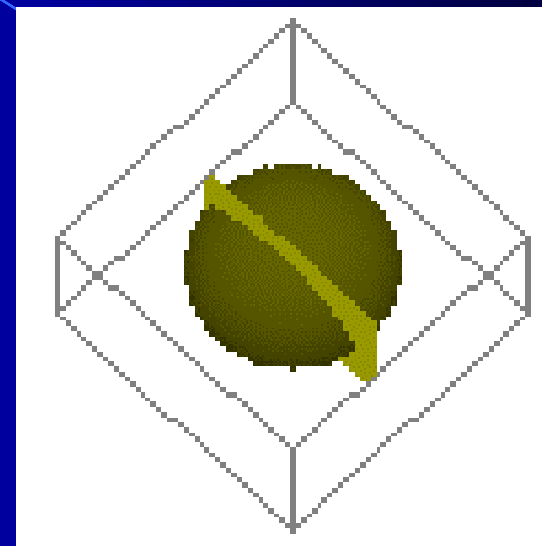
Boundary Genus = 0

Focus of the Genus for boundary of 3D Objects (Raster Space)



M3= 8 ; M4= 4198 ; M5= 24; M6= 0

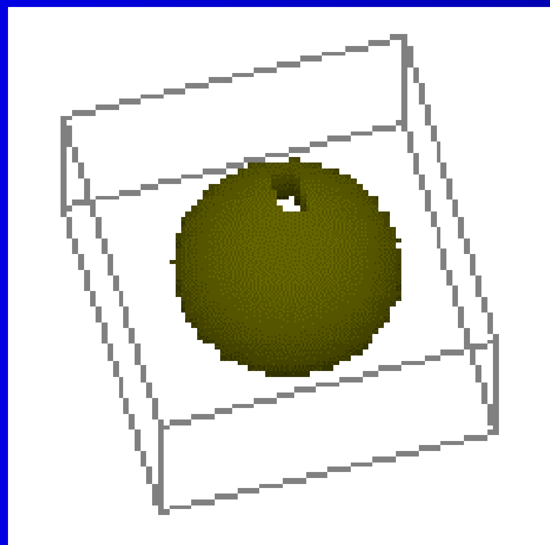
Boundary Genus= 3



M3= 2236 ; M4= 4862; M5= 1508 ; M6= 360

Boundary Genus = 0

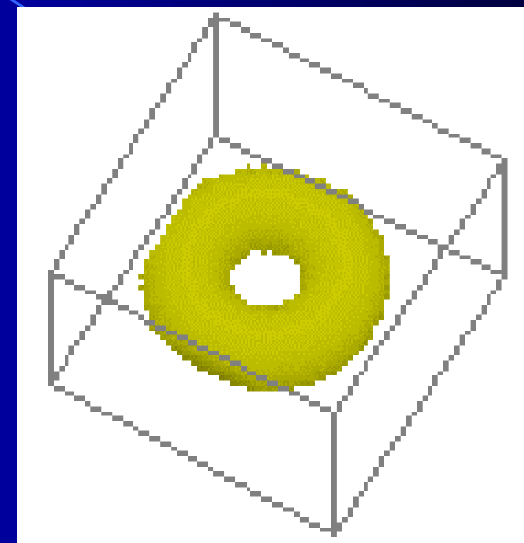
Focus of the Genus for boundary of 3D Objects (Raster Space)



M3= 2204 ; M4= 4236; M5= 1484 ; M6= 360

Boundary Genus= 1

totalNoneM5 0, totalNoneM6 0

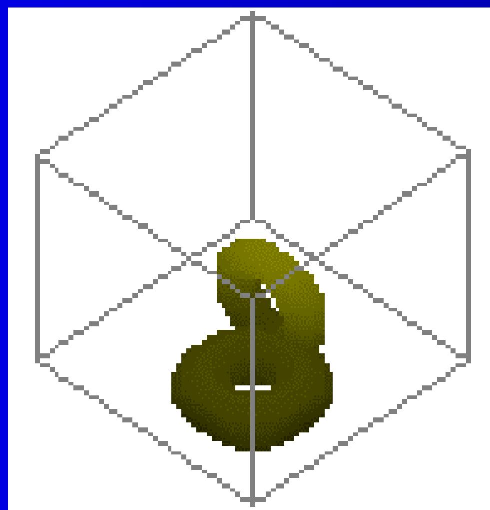


M3= 1792 ; M4= 3600 ; M5= 1248 ; M6= 272

Boundary Genus = 1

totalNoneM5 0, totalNoneM6 0

Focus of the Genus for boundary of 3D Objects (Raster Space)

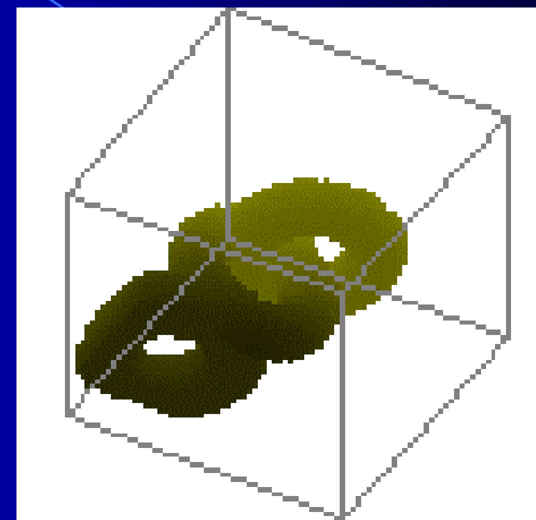


M3= 1276 ; M4= 2974 ; M5= 876 ; M6= 204

Boundary Genus= 1

totalNoneM5 0, totalNoneM6 0

manifold Genus= 2



M3= 2268 ; M4= 5456 ; M5= 1688 ; M6= 301

totalNoneM5 2, totalNoneM6 2

No result since 4 pathological situations are included in the objects.

Must delete those points in order to calculate the genus correctly.

References

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