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λ-Connectedness and Its Applications

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Abstract: This paper attempts to provide a systematic view to the λ-connectedness method. This method is for classification/segmentation, fitting/reconstruction, and inference. A common type of partial connectivity that describes the phenomenal of gradual variation is studied in various domains. The previous research work on λ-connectedness was abstracted and integrated into a unified framework: a network- or graph-based system. This method focuses on a way to solve a set of problems that λ-connectedness can apply to rather than to give a solution to a particular problem. This technique is based on a graph $G = (V, E)$ and an associate function ρ on the vertices of the graph, where ρ is called the potential function. A measure $C_\rho(x, y)$ is defined for the λ-connectedness on the vertices $x, y \in G$ with respect to ρ. For a certain $\lambda \in [0, 1]$, x and y are said to be λ-connected if $C_\rho(x, y) \geq \lambda$. So, λ-connectedness is a type of fuzzy relation. If every pair of vertices are λ-connected, then ρ is λ-connected on G. Therefore, λ-connectedness is also a measure of continuity in discrete spaces and systems. Based on this concept, the relationship between graph theory and λ-connectedness is discussed, and the λ-connected segmentation and fitting are introduced. Further more, the fast graph-theoretic algorithms to problems of λ-connectedness are described. At the end, the relationship among λ-connectedness, rough sets, belief networks, and network economics and resource management are also investigated. The λ-connectedness method can be viewed as a special application of graph theory, a fuzzy system method, or a systematic method for problems in uncertainty.

Keywords λ-Connectedness, graph, image segmentation,surface reconstruction,uncertainty

1. Introduction

Connectedness is a basic measure in many research areas of mathematical science and social sciences. In graph theory, two vertices are said to be connected if there is a path between them. [31] In topology, [30] two points a and b are connected in space S if there is a continuous function $f : [0, 1] \to S$ such that $f(0) = a$ and $f(1) = b$. In management science, two individuals are connected, for example, in an institution if one person is under the supervision of the other. Such connected relations only describe either full connection or no connection. λ-connectedness is introduced to measure incomplete or fuzzy relations between two vertices, points, human beings, etc.

In fact, partial relations have been studied in other aspects. Random graph theory allows one to assign a probability to each edge of a graph. [1] This method assumes, in most cases, each edge has the same probability. On the other hand, Bayesian networks, are often used for inference and analysis when relationships between each pair of states/events, denoted by vertices, are known. These relationships are usually represented by conditional probabilities among these vertices and are usually obtained from outside of the system. [25] .

A type of fuzzy connectivity was studied by A. Rosenfeld in 1979 in his pioneer paper on fuzzy digital geometry and topology. [40] Rosenfeld treated a digital image as a 2D fuzzy set. A threshold can be set to determine a component that is connected (in domain) and each point has the value above the threshold. When the threshold changes the segments changes. It is called fuzzy connected components. Chen in 1985 suggested another type of fuzzy connectivity by observing the properties of seismic velocity layer in geophysics. [2] What often makes a layer is the similar intensity value in a seismic image. The velocities of two locations can have a relatively big difference vertically in a layer as long as they can gradually change from one to another. The standard segmentation method using statistical mean as the uniformity will not fit to this situation. He defined a similarity measure for two pixels that was called *lambda*-connectedness. In 1986, in order to measure the continuity of a component in an image. Rosenfeld defined "discretely continuous" on gray scale images. Two adjacent pixels are "discretely continuous" if the difference of the two intensity values is less than 1. The concept of "discretely continuous" was called gradually varied by Chen in 1989. [3] Chen proved the theorem for gradually varied extension. In fact, *lambda*-connectedness, discrete continuous, and gradual variation have the same philosophy, but *lambda*-connectedness can be more general and cover other two as the special cases. Pawlak used rough continuous that can also be a special case of λ-connectedness. [37]

λ-connectedness is built on a system with a general graph and an attaching a function to vertices of the graph. After that one can study partition, smoothness, reconstruction and other purposes. the inference of the function on the graph–the general domain of the function. Many real problems can be deduced to this model. It has deep relationship to connectivity and continuity.

The new development on λ-connectedness has made some progress in both theory and practice. We have developed new methods for practical data reconstruction and fitting. [13] We have combined λ-connected segmentation with the wavelet method in finding region outliers in Meteorological Data. [32] We also discuss how to determine an appropriate λ value for segmentation. [12] We also used it for bone density research [11]. Then we have discussed how do we use *lambda*-connectedness for very fast algorithm for compressed internet images using the qardtree images. [?]

This paper was written based on Chen's PhD thesis of University of Luton and several other articles. [14] The earlier version of this paper was used in Chapter 10 of Chen's book. [10] It was modified recently by adding some new research attempts, and we also discuss how to use *lambda*-connectedness technology in practical problems.

2. A Simple Example of λ-connectedness

Chen [2,?] and Chen *et al.* [15,?] proposed λ-connectedness to describe the phenomenon that geophysical and geological parameters and properties exhibit gradual or progressive changes in a layer, but sudden changes frequently occur between two layers. [18] (See Fig.1) In other words, a λ-connected set can be viewed as a layer, and two layers should be separated by λ-connectedness.

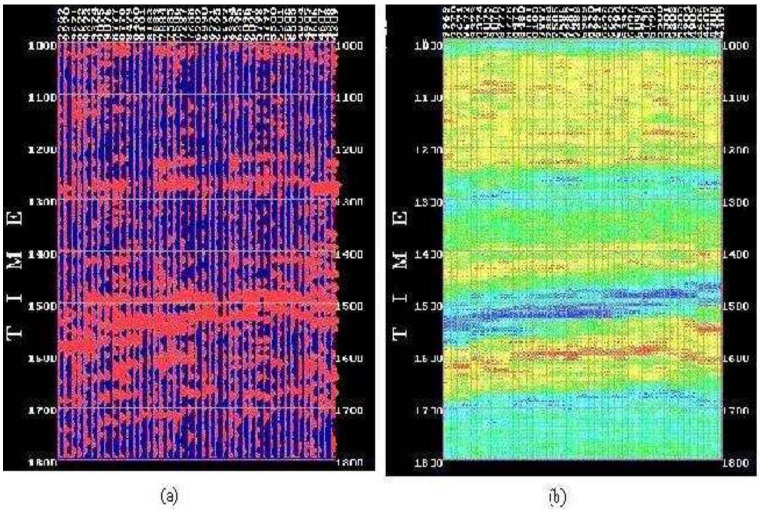

Fig. 1. Example of seismic data: (a)the wave form data, (b) A type of velocity data

An image is a mapping from a two dimensional space to the real space R. Without loss of generality, let Σ_2 be the two-dimensional grid space, we call the 2D digital space. A digital image can be represented by a function: $f : \Sigma_2 \to [0, 1]$. Let $p = (x, y), q = (u, v) \in \Sigma_2$, p, q are said to be adjacent if $\max\{\|x-u\|, \|y-v\|\} \le 1$. (A pixel, i.e. picture element, is a couple of $(p, f(p))$.)

So, if p, q are adjacent and $f(p), f(q)$ only have a "little" difference, then pixel $(p, f(p))$ and $(q, f(q)$ are said to be λ-adjacent. If there is point r that is adjacent to q and $(q, f(q)), (r, f(r)$ are λ-adjacent, then $(p, f(p)), (r, f(r)$ are said to be λ-connected. Similarly, we can define the λ-connected on a path of pixels.

Assume that there is a 4×4 small image, all pixels $p_{i,j}$, $i, j = 1, 2, 3, 4$, are given in the following array.

$$\begin{bmatrix} .2 & .1 & 1. & .8 \\ .1 & .8 & .9 & .5 \\ .7 & .8 & .4 & .6 \\ .7 & .4 & .6 & .8 \end{bmatrix} \qquad (1)$$

One can see, if the "little" difference is set to be 0.2, then two pixels are λ-adjacent if the difference between two adjacent elements are not greater than 0.2. It is easy to see that there are three λ-connected components in the image. The second component is shown below:

$$\begin{bmatrix} & & 10. & 8.0 \\ & 8.0 & 9.0 & \\ 7.0 & 8.0 & & \\ 7.0 & & & \end{bmatrix}. \tag{2}$$

The other two components are located at the up-left corner and the low-right corner of the original matrix (1), respectively.

Mathematically, let (Σ_2, f) be a digital image. If p and q are adjacent, we define a measure called "neighbor-connectivity" below:

$$\alpha_f(p,q) = \begin{cases} 1 - \|f(p) - f(q)\|/H & \text{if } p \text{ and } q \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where $H = \max\{f(x)|x \in \Sigma_2\}$.

Let $x_1, x_2, ..., x_{n-1}, x_n$ be a simple path. The path-connectivity β of a path $\pi = \pi(x_1, x_n) = \{x_1, x_2, ..., x_n\}$ is defined as

$$\beta_f(\pi(x_1, x_n)) = \min\{\alpha_f(x_i, x_{i+1})|i = 1, ..., n-1\} \tag{4}$$

or

$$\beta_f(\pi(x_1, x_n)) = \prod\{\alpha_f(x_i, x_{i+1})|i = 1, ..., n-1\} \tag{5}$$

Finally, the degree of connectedness (connectivity) of two vertices x, y with respect to ρ is defined as:

$$C_f(x,y) = \max\{\beta_f(\pi(x,y))|\pi \text{ is a (simple) path.}\} \tag{6}$$

For a given $\lambda \in [0,1]$, point $p = (x, f(x))$ and $q = (y, f(y))$ are called λ-connected if $C_f(x,y) \geq \lambda$.

If equation (4) applies, λ-connectedness is reflexive, symmetric, and transitive. Thus, it is an equivalence relation. If equation (5) is used, λ-connectedness is reflexive and symmetric. Therefore, it is a similarity relation. [2, 4, 15, 16, 31]

3. Generalized λ-Connectedness

This paper integrates several λ-connectedness based techniques into a simple system $< G, \rho >$, where ρ is called a potential function of G. If $< G, \rho >$ is an image, then G is a 2D or 2D grid space and ρ is an intensity function. For a color image, we can use $f = (red, green, blue)$ to represent ρ. If $< G, \rho >$ represents the relationships among companies, then each vertex represents a company and each edge represents that there exist business transactions between these two companies. For example,

$\rho(v)$ can represent products and revenues for the company v, where v is a vertex in G. In management science, $< G, \rho >$ could represent the individual relationships. Suppose G is a directed graph in this case: each vertex represents a person, an edge(arc) represents a supervisory relationship, and ρ represents the duties and powers in a company. ρ may also be a vector in which each component of the vector indicates a person's responsibilities on a specific aspect.

3.1. *λ-connectedness on Undirected Graphs*

λ-connectedness can be defined on an undirected graph $G = (V, E)$ [31] with an associated (potential) function $f : V \to R^m$, where R^m is the m-dimensional real space. Given a measure $\alpha_\rho(x, y)$ on each pair of adjacent points x, y based on the values $\rho(x), \rho(y)$, we define

$$\alpha_\rho(x, y) = \begin{cases} \mu(\rho(x), \rho(y)) & \text{if } x \text{ and } y \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where $\mu : R^m \times R^m \to [0, 1]$ with $\mu(u, v) = \mu(v, u)$ and $\mu(u, u) = 1$. Note that one can define $\mu(u, u) = c$ and $\mu(u, v) \leq c$ where $c \in [0, 1]$ for all u. α_ρ is used to measure "neighbor-connectivity." The next is to develop path-connectivity so that λ-connectedness on $< G, \rho >$ can be defined in a general way.

In graph theory, a finite sequence $x_1, x_2, ..., x_n$ is called a path, if $(x_i, x_{i+1}) \in E$. A path is called a simple path if $x_i \neq x_j$, $i \neq j$ excepting $x_1 = x_n$. The path $x_1, x_2, ..., x_{n-1}, x_n = x_1$ is called a cycle. The path-connectivity β of a path $\pi = \pi(x_1, x_n) = \{x_1, x_2, ..., x_n\}$ is defined as

$$\beta_\rho(\pi(x_1, x_n)) = \min\{\alpha_\rho(x_i, x_{i+1}) | i = 1, ..., n-1\} \tag{8}$$

or

$$\beta_\rho(\pi(x_1, x_n)) = \prod\{\alpha_\rho(x_i, x_{i+1}) | i = 1, ..., n-1\} \tag{9}$$

Finally, the degree of connectedness (connectivity) of two vertices x, y with respect to ρ is defined as:

$$C_\rho(x, y) = \max\{\beta(\pi(x, y)) | \pi \text{ is a (simple) path.}\} \tag{10}$$

For a given $\lambda \in [0, 1]$, point $p = (x, \rho(x))$ and $q = (y, \rho(y))$ are said to be λ-connected if $C_\rho(x, y) \geq \lambda$. In image processing, $\rho(x)$ is the intensity of a point x and $p = (x, \rho(x))$ defines a pixel. (Note: We usually avoid to say that x and y are λ-connected because it does not contain the information of ρ. We may say two pixels are λ-connected, but generally not that two points are λ-connected.)

If $< G, \rho >$ is an image, then this equivalence relation can be used for segmentation meaning to partition the image into different objects. On the other hand, if a potential function f is partially defined on G, then one can fit f to be ρ such that $< G, \rho >$ is λ-connected on G .

3.2. λ-connectedness on Directed Graphs

The only difference between a directed graph $G = (V, A)$ and an undirected graph $G = (V, E)$ is that if $(x, y) \in E$ then $(y, x) \in E$ whereas if $(x, y) \in A$ it may or may not be true that $(y, x) \in A$. So, for λ-connectedness, there may be a case that p can be λ-connected to q but q cannot be λ-connected to p. For example, an irrigation systems can be described on a directed graph. Suppose there are N water reservoirs in the system, then let x be a reservoir and $\rho(x)$ be the amount of energy/power of x. If $(x, \rho(x))$ is λ-connected to $(y, \rho(y))$, then reservoir x can transfer water to reservoir y. It may not be the case that $(y, \rho(y))$ can be λ-connected to $(x, \rho(x))$.

Similarly, define α_ρ and β_ρ on $< G = (V, A), f >$,

$$\alpha_\rho(x, y) = \begin{cases} \mu(\rho(x), \rho(y)) & \text{if arc (x, y) is in A} \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

In a directed graph, a finite sequence $x_1, x_2, ..., x_n$ is called a strongly connected path if $(x_i, x_{i+1}) \in A$ for all $i = 1, ..., n-1$. The sequence is called weakly connected if $(x_i, x_{i+1}) \in A$ or $(x_{i+1}, x_i) \in A$ for all i. The path-connectivity of a path $\pi = \pi(x_1, x_n) = \{x_1, x_2, ..., x_n\}$ is still defined as

$$\beta_\rho(\pi(x_1, x_n)) = \min\{\alpha_\rho(x_i, x_{i+1}) | i = 1, ..., n-1\} \tag{12}$$

or

$$\beta_\rho(\pi(x_1, x_n)) = \prod \{\alpha_\rho(x_i, x_{i+1}) | i = 1, ..., n-1\} \tag{13}$$

Thus, the degree of connectedness (connectivity) is defined as

$$C_\rho(x, y) = \max\{\beta(\pi(x, y)) | \pi \text{ is a (simple) path }\} \tag{14}$$

where the path can be a strongly connected path or a weakly connected path. A weakly connected path appears to be the same as a path in an undirected graph. In directed graphs, a path usually means a strongly connected path.

We should note that λ-connectedness on undirected and directed graphs apply to different problem categories. The former mainly deals with the similarities of two individuals on a "network." The later is for "sending," "receiving" and "finding." In most applications, an acyclic directed graph is required for simplifying analysis procedures. An acyclic graph means that there is no (strongly connected) cycle in the graph. [1, 31]

3.3. Potential Function ρ and the Measure μ

For λ-connectedness, the potential function ρ, and μ are two key factors. If G is a lattice or a ϵ-net [10], ρ may be a polynomial function. For a color image, ρ is the $RGB = (r, g, b)$ function. For a resource network, $\rho(x)$ can represent the utility function or resource potentials of the nodes in multi-dimensions. For example, a city that utilizes high electricity, generates more jobs, and has many universities can be represented by the following utility vector

$$\rho(theCity) = (highElectricity, moreJobs, manyUniversities, ...). \quad (15)$$

Measure μ may also be vary depending on different situations. In previous work, three kinds of μ were used [2,4,15]:

$$\mu_1(u,v) = \begin{cases} 1 - \frac{\|u-v\|}{\|u\|+\|v\|} & \text{if} \|u\| + \|v\| \neq 0 \\ 1 & \text{otherwise} \end{cases} \quad (16)$$

and

$$\mu_2(u,v) = 1 - \frac{\|u-v\|}{H}, \quad (17)$$

where $H = max\{\|u\| | u \in \rho(V)\}$. The third kind of μ is defined by a so-called gradual variation relation of a sequence of real (or rational) numbers $A_1 < A_2 < ... < A_n$. Here, $\rho : V \to \{A_1, A_2, ..., A_n\}$ for $G = (V, E)$ (or $G = (V, A)$). Let $u = A_i$ and $v = A_j$, $\mu(u,v) = 1$ if $i = j - 1$, j or $j + 1$; otherwise $\mu(u,v) = 0$. Thus, λ-connectedness based on gradual variation can be defined: [3,7] with the value of λ being "1." The gradually varied function ρ plays a key role in λ-connected fitting, which will be discussed in detail later. Rosenfeld used a similar concept called the "continuous" digital function for image processing. [41]

3.4. *Theorems for Partition and Extension*

As most engineers known, a complete view of the Fourier transformation has two parts: transform and inversed transform. For λ-connectedness, we shall have two opposite directions: partition and interpolation-extension. The following theorems guaranteed that the λ-connected method is a reasonable approach.

Theorem 1 (The Decomposition Theorem): Given a value of λ, all λ-connected components form a partition of the vertices of the graph.

Theorem 2 (The Extension Theorem) : Assign values on a subset of the vertices, the necessary and sufficient condition for the existence of a λ-connected extension (to the whole vertex set) is that there is a valuation on each shortest path in the graph such that the path is λ-connected.

4. Methodology Relates to *lambda*-connectedness

The λ-connectedness method provides (Theorem 1) an approach to unsupervised classification, i.e. clustering. The potential function can be viewed as a feature vector. Unlike the typical problem of clustering, domain will be countered as a factor. The domain has a structure, not just the individual elements of a general set. For instance, separating a pail of apples into several groups does not need λ-connectedness. However, λ-connectedness can apply to separate habitation and forests from satellite images since the domain is geographical region.

The λ-connectedness method can also apply to fitting and data reconstruction (Theorem 2) for arbitrary domain. However, it cannot be directly used for high degree smooth surface fitting. For example, if one asks to fit a surface of the front of an automobile, the λ-connectedness method could not directly make it. When we try to fit an interface of a geological layer, the smoothness is not a big issue anymore, then we can use the λ-connectedness method.

However, why the λ-connectedness method is necessary for some specific problems? We will discuss them in the rest of this section.

When a problem occurs, (G. Polya's four phase) Polya suggested four steps to solve the problem: (1) Understand the problem (analysis), (2) devise a way to solve the problem (algorithm design/system design), (3) implementation, and (4) test the results. This section will discuss what the λ-connectedness method do and when we should selected it to solve a problem, and how to solve a problem.

4.1. *Analysis: General Characteristics of the problem and solution*

In order to use the λ-connectedness method appropriately, we need to analyze the suitable cases.

First, the problem has a domain structure, the more complex the better fit. Because the existing method can deal with regular domain better.

Second, the solution contains the large amount of samples.

Third, there is no explicit mathematical equation to describe the problem. Some rough solution is acceptable.

Data size is relatively big. fast processing is critical. Otherwise, we can search for every possibility.

The solution containing samples has the property of continuity or gradual variation.

Case study 1: regional economics vs. stock marketing. If one was asked to build a model to describe a special aspect in regional economics. We can consider to use the λ-connectedness method without doing detail investigation since regional economics would not be jumped in days. The domain is a network can not be growing by a city along. The domain is irregular. If the solution set meets the λ-connectedness, we should definitely try to use it at least in some way. However, we would like to reject immediately to use the λ-connectedness method to predict stock market since even the domain is in time the curve could jump several times a day.

4.2. *Design: Detail steps to use the λ-connectedness method*

Once one decides to try to use the λ-connectedness method.

First, you want to investigate what type of domains (the place to store a problem mathematically) that you want to select. Sometimes it is not obvious. In image processing, a rectangle domain is very popular to use. However, in some cases, an image is stored in a quadtree mode. Do you always want to restore image into a rectangle domain then do analysis? A graph can be generated for this case and λ-connected search can be done in a very fast way. We will discuss more in Section 5.4.

Second, select the connectivity function. It depends on the problem you are trying to solve. Some general requirements were discussed in papers. [4, 15]

Third, algorithms design including how to use the existing standard algorithms such as depth-first search or breadth-first search. [21] Implementation including coding is to choose a software package or code by users. To test the result is also important for an appropriate use of λ-connectedness. (If there are some standard methods that are better than the λ-connectedness method, one can choose the better method.)

Based on the two main theorems (Theorem 1 and 2), the task that the λ-connectedness method can do has the following characteristics: (1)find a similar parts (clustering), and (2) insert a missing parts (interpolation). The first can be a decision problem; the second is a prediction.

λ-connectedness could provide a rough solution. When the detail properties are known, then λ-connectedness might not be a good choice.

For example, a camera captured a few pictures for a flying object. The question is to determine where it comes from and where it goes? There are two possibilities:(1) the object has no engine (2) the object has an engine. For (1) we know the trace of object will be followed by a parabola, we only need to fit the formula. However if we choose λ-connectedness, then may predict an answer if there are enough points are captured. but it may produce a very fake answer. for (2) we really have no clue to find where the object original come from but can get some idea on how it travels in the period of captured pictures and some idea on λ-connectedness may be applied to find speed and possible next position. Since it is not possible to get a precise solution, the result could be wrong but we are aiming a reasonable answer.

Again, a function has two parts: its domain and range. λ-connectedness is defined on graphs; however graph theory only deals with vertices and edges with or without weights. In order to define a partial, incomplete, or fuzzy connectedness, we need to assign a function on the vertex in the graph. Such a function is called a potential function. It can be used to represent the intensity of an image, the surface of a XY-domain, or the utility function of a management or economic network.

4.3. *Some Application Remarks*

This subsection presents some additional information to Introduction of this paper. In 1994, Chen, Berkey and Johnson formulated a technique for ionogram scaling based on a λ-connected search. [15] Tsai and Berkey developed a software system to do real-time ionogram scaling based on a fuzzy λ-connected search. [45] In 1998, Chen *et al* applied λ-connected fitting in the reconstruction of seismic volume data. [17] In 1994, Chen, Cheng and Zhang expanded the concept of λ-connectedness to multi-dimensional range images in digital spaces. [16] They called such a space a fuzzy sub-fiber. Tsai, Berkey, and Liu found a special connected function in fuzzy sub-fibers in ionogram scaling. [46]

Based on the concept of λ-connectedness, we introduce λ-connected segmentation for 2D or 3D images. Two types of λ-connected segmentation methods will be discussed. The first one uses region growing mechanism and the second one is called λ-connected split-and-merge segmentation. [4]

Given some guiding points, i.e. some vertices with values of a potential function ρ. One needs to to find the ρ for all vertices that are λ-connected. Such a problem is called the problem of λ-connected fitting. We know that for each special function C_ρ the answer may be different. However, for a heavily used C_ρ and the discritization of the range(-domain)–gradually varied functions, the theorem for the necessary and sufficient condition is proven. [3] The approximation problem in relation to optimum fitting was discussed, and real examples related to intelligent fitting were also considered. [?, 7]

We also studied the relationship between graph theory and λ-connectedness. [10] Given a potential function ρ, if we are only interested in the connections among the vertices, then the problem can be translated to a special weighted graph where each weight on edges can be calculated based on ρ. Many existing fast graph algorithms can be directly used to solve problems related to λ-connectedness, such as maximum connectivity spanning tree and the maximum connectivity between two points/vertices.

Other applications we discuss in this paper will be its relationship with rough sets, belief networks, and its potential uses in network economics and resource management. We also present some applications using the technique in curve searching, data mining, and management science.

Rough sets was first studied by Pawlak to describe the approximation of a set by using lower and upper bound of the set. A relationship between rough calculus and gradually varied function was addressed by Pawlak. [37] In addition, the relationship between λ-connectedness and rough sets was studied by Chen. [9]

This paper demonstrates another relationship in which two λ-connected sets is used to represent the lower bound and the upper bound of a rough set. Therefore, the determination of a rough set is equivalent to finding two λ-connected sets. This transformation allows us to directly use fast algorithms for problems related to data mining.

Belief networks, also known as Baysian networks, [25] use directed graphical models that often deal with influence diagrams. Undirected graphical models have simpler conditional independence that are referred as Markov random fields. The relationship between two nodes (vertices) in graphical models is described by an weighted edge, and its weight is represented by a probability value. The advantage of using a Baysian network is that it performs a precise analysis. However, it is not efficient for complex systems, and it requires extensive knowledge. For example, if one uses Baysian networks to process hundred-thousands of image pixels, it may take hours or even days to segment the image. On the other hand, if one uses a λ-connected flow to find the cause of a disease for an individual, it is not appropriate since it may be too rough. λ-connected flow may be appropriate for a large group of populations.

Management or economic networks are very popular today in modern economics. [23] We have included several real problems which have connections with λ-connectedness. Although most of the work related to λ-connectedness uses an undirected graph, this paper will present the concept for directed graphs.

5. λ-connectedness and Image Segmentation

Segmentation is a clustering method used in image processing. [22, 36] There are many kinds of segmentation, e.g. measurement space clustering, region growing, split-and-merge segmentation, edge detection, etc. Different types of data require different segmentation methods.

For gray-scale images such as velocity profiles, we can use region growing or edge detection. For texture images such as waveform profiles or seismic sections, one may need rule-based segmentations.

As we know, a digital image F is a mapping from a grid-space Σ to the real set R (or R_n in general). Let S be a connected component of Σ. S is said to be uniform if F on S (a sub-image) has properties of uniformity. For instance, a popular uniformity measure is defined in terms of the maximum difference between any pixel value and the mean value of S.

Segmentation partitions an image into connected subsets (segments). Each segment is uniform, and no union of adjacent segments is uniform. The formal definition of segmentation is then: In a digital image F, if there exists a non-empty segmentation $F_1, F_2, ..., F_m$, satisfying: (1) $F_i \cap F_j = \emptyset$, if $i = 1, .., m, j = 1, ..m, i \neq j$, (2) $\cup_{i=1,...,m} F_i = F$, (3) each F_i is connected, (4) each F_i is "uniform," and (5) if F_i and F_j are adjacent, then $F_i \cup F_j$ is not uniform, then, $\{F_1, F_2, ..., F_m\}$ is called a segmentation of F. In this paper, we describe two geometrical segmentation methods, region growing and split-and-merge segmentation, and their relationship to λ-connectedness .

5.1. *λ-connected Region Growing Segmentation*

In a gray scale image, intensity is the uniformity measure. A region (or segment) in an image may be viewed as a connected group of pixels, all with similar brightness. The region growing method begins with a single pixel, and then by examining its neighbors tries to find a maximum sized connected region of similar pixels. In this manner regions grow from single pixels. One also can use a region or grouped set of pixels as a seed instead of a single pixel. In this case, after selecting the partition (group of pixels), a uniformity test is applied to the region to see if it qualifies as a partition. If the test fails the region is subdivided into smaller regions. This process is repeated until all regions are uniform. (The major advantage of using small regions rather than single pixels is that it reduces the sensitivity to noise.)

Region growing forms an equivalence relation to partition the image. λ-connected segmentation is used to partition the image by searching each λ-connected component in the image. The fast algorithm design technique such as depth-first-search or breadth-first-search can be used for implementation [36]. This method has been applied to several applications such as seismic layer search [18] and ionogram scaling [15, 45, 46]. The algorithm will be discussed in Section 8.

5.2. *λ-connected Split-and-Merge Segmentation*

In two dimensional image processing, split-and-merge segmentation is based on a quadtree partition of an image and hence is sometimes called quadtree segmentation. One starts with tree nodes (representing square regions of the image) at some intermediate level of the quad-tree. If a square is found to be non-uniform, then it is replaced by its four son-squares (split). Conversely, if four son-squares are segmented and a region in a son-square can merge with a region in another son-square in terms of adjacency and uniformity, they will be merged until no more pair of regions that can be merged. These four son-squares are replaced by a single square(merge), called a segmented square. This process continues recursively until no further splits or merges are possible.

In traditional split-and-merge segmentation, the uniformity is measured by the mean of the merged region. This is a statistical measure. In λ-connected split-and-merge segmentation, we merge two regions into one if the merged region is λ-connected for any possible path in the region. Such a region is called a normal λ-connected set [4, 10] Split-and-merge segmentation only preserves reflexivity and symmetry and is not a mathematical partition or equivalence classification.

5.3. *Comparison between the λ-connected method and the traditional segmentation method*

Region growing method was referred to be the component labeling method in Rosenfeld and Kak's book and Pavlidis's book. [36, 42] The technique used was only for binary images. If we set $\lambda = 1$, then λ-connected segmentation will get the same result as the component labeling method does. A more general consideration described in Gonzales and Wood's book [22] is similar to the example in Chen and Adjei's paper. [13] Therefore, the methods described in these books [22, 42] are the special cases of λ-connected region growing segmentation, so it can be viewed as a generalized region growing Method.

On the other hand, using λ-connected segmentation, one could avoid the errors made by converting a gray-scale image to a binary image. For example, if we want convert a gray-scale image in which the pixel with the intensity between 10 to 20 will be set to "1", otherwise "0". If there three pixels in a row, $P_{11}P_{12}P_{13}$ where $P_{11} = 9$, $P_{12} = 10$, and $P_{13} = 20$. What we did was to treat that the pixel with the intensity "10" and the pixel with the intensity "20" are in the same object and cut out P_{11}. This is not reasonable in many cases since the value distance between P_{11} and P_{12} is only one. Using λ-connectedness, this converting process is not necessary.

For split-and-merge segmentation, the λ-connectedness method is good for certain situations. As we described in Section 5.2, the λ-connected split-and-merge segmentation finds the part in which a pixel must λ-connected with all of its neighbors. In other words, this part must be "continuous." Rosenfeld discussed the "continuous" in digital pictures [41] where he treated the continuous as the level of difference to be not greater than "1" (or a certain number in general). We can see that λ-connectedness in this case is more general. In the next section, we will present a technique to interpolate a discrete "continuous" function based on this concept.

5.4. *λ-connected segmentation for quadtree represented images*

In above applications, the domain is regular i.e. a rectangle region. If an image is compressed by quadtree technique, we do not have to restore /decode the image instead of directly using the quadtree patition and the graph built by the quadtree to perform λ-connected segmentation.

This algorithm will be described below:

The advantage is the speed. Typical Image segmentation must go through each point so the time complexity must be at least $O(n^2)$ n is the length of the image and $n = 2^k$. let the number of quadtree leaves is N. we first need to get the adjacency graph $G_Q = (V_Q, E_G)$ of the quadtree stored image.

This graph has $N = |V_Q|$ nodes, and at most $3N$ (or $4N$ for the 8-neighbor system) edges. Let's verify this Proposition.

Since each node in G_Q is a quadtree leaf or a 2^t,$t \leq k$ square in the original image. Each pixel in the square has the same (or nearly the same for loss-representation) value. In other words, G_Q provides a partition for the original image $< G, \rho >$. We define that a edge of the partition is at least a side of a leaf. The vertex of the partition is at least a corner point of leaf. Both edges and vertices are virtual to the image. We have calculated that we only need a $O(|V_Q|)$ time algorithm to perform the segmentation using $lambda$-connectedness. The value of $|V_Q|$ is usually much smaller than n^2 the original image size. Even through $|V_Q|$ is dependent on the actual image, but it is very reasonable to say that the average is $O(n)$. [10, 19]

Without decode the image one cannot perform a statistical mean-based segmentation since it is not a math classification, a big block to be added to a segment probably not satisfy the requirement of $|p - m| < \delta$. It needs to break a leaf to get the more precise segmentation.

5.5. *Determination of λ values*

It is a natural and unavoidable question how we determine λ value in connectedness-based segmentation? It is somehow similar to determine the clip level in threshold segmentation; however, $lambda$ value is not as sensitive as the clip-level for threshold. There are fewer λ values to be selected than clip-levels. In Chapter 10 of Chen's book [10], Chen provided the detail analysis on this issue.

Some techniques have been proposed and tested such as the binary search-based method and the maximum connectedness spanning tree method [10] [9].

Several methods have been proposed recently for finding a reasonable value for λ based on the specific applications.

In this subsection, we mainly introduce a method that uses the maximum entropy method to determine the λ value [12]. Other methods such as the method relating to minimum variance can also be found in Chen's paper [12]. The maximum entropy method was first proposed by Kapur, Sahoo, and Wong [26]. It is based on the maximization of inner entropy in both the foreground and background. The purpose of finding the best threshold is to make both objects in the foreground and background, respectively, as smooth as possible. [26, ?, ?]

If F and B are in the foreground and background classes, respectively, the maximum entropy can be calculated as follows;

$$H_F(t) = -\Sigma_{i=0}^{t} \frac{p_i}{p(F)} \ln \frac{p_i}{p(F)} \tag{18}$$

$$H_B(t) = -\Sigma_{i=t+1}^{255} \frac{p_i}{p(B)} \ln \frac{p_i}{p(B)} \tag{19}$$

where p_i can be viewed as the number of pixels whose value is i; $p(B)$ is the number of pixels in background, and $p(F)$ is the number of pixels in foreground. The maximum entropy is to find the threshold value t that maximizes $H_F(t) + H_B(t)$.

In λ-connected segmentation, we can also use this idea. The total inner entropy for the image is to calculate the entropy for each λ-connected component, not for the threshold clipped foreground/background. This is because in λ-connected segmentation there is no specific background. Each λ-connected segment can be viewed as foreground, and the rest may be viewed as the background. It is different from the original maximum entropy where the range of pixel values determines the inclusion of pixels. (One needs to summarize all inner entropies in all segments.)

$$H(\lambda) = \Sigma(\text{inner entropy of each } \lambda\text{-connected component}) \tag{20}$$

We will select the λ such that $H(\lambda)$ will be maximized. This unique value is a new measure for images. Since the maximum entropy means the minimum amount of information or minimum variation, we want the minimum change inside each segment. We use the λ_e such that

$$H(\lambda_e) = \max\{H(x) | x \in [0,1]\}. \tag{21}$$

Assume there are m λ-components, define inner entropy of each λ-component S_i:

$$H_i(\lambda) = \Sigma_{k=0}^{255} - \frac{Histogram[k]}{n} \log \frac{Histogram[k]}{n} \tag{22}$$

where n is the number of points in the component S_i. $Histogram[k]$ is the number of pixels whose values are k in the segment. Thus,

$$H(\lambda) = \Sigma_{i=1}^{m} H_i(\lambda)) \tag{23}$$

The maximum entropy connectedness can be viewed as a measure of a special connectivity for the image. If λ value is calculated in the above formula for an image that makes $H(\lambda)$ to be maximum, we call that the image have the maximum entropy connectedness λ, denoted as λ_e.

In 2006, [11] we proposed a golden cut method for finding the λ-value for bone density connectedness calculation. We have obtained a λ=0.96, 0.97 for a bone image (the size of the picture is different from the one used in this paper). For a similar image, using the maximum entropy connectedness presented in this section, we got λ_e=0.95. The result is quite reasonable. The original image and both of the segmented images are shown in Fig. 2-5. No pre-cut (preprocessing) is performed in the segmentation.

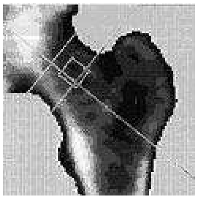

Fig. 2. Bone Density Image Segmentation : the Original image

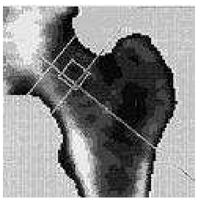

Fig. 3. Bone Density Image Segmentation : λ=0.97

What we can see in the above three segmented images (Fig. 3-5). Fig. 3 seemed to be the same as the original image Fig. 2. It is possible that Fig. 5 may represent the better understanding of bone connectivity. However, what we state here is that λ_e can provide us the meaningful result and it was done automatically.

For the commonly used testing image "Lena," Fig.6, the result of λ-connected segmentation is quite interesting. Whatever we use a pre-threshold cut or not, λ_e is

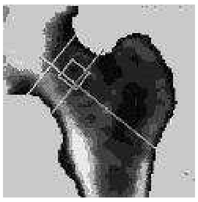

Fig. 4. Bone Density Image Segmentation : λ_e=0.95

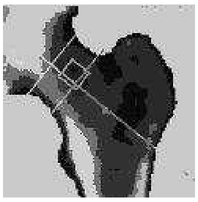

Fig. 5. Bone Density Image Segmentation : λ=0.93

always 0.99 (Fig. 7). An original maximum entropy arrived at the clip-level of 125 counts of the 8-bit gray level image (0-255 of the pixel value range. The reason is that the "Lena" image does not contain many "continuous" parts. In the λ-connected segmentation, we can see that λ_e (=0.99) connected segmentation has connected the continuous component especially at the face and shoulder. This matches the result of using the maximum entropy cut, Fig. 8 (We use NIH *ImageJ* to perform the cut.) Thus, we can say that our new method is still reasonable. When we use $\lambda = 0.98$ for the image, we get Fig. 9.

Fig. 6. Image Segmentation for testing image "Lena:" the Original image

Fig. 7. Image Segmentation for testing image "Lena:" λ=0.99

Fig. 8. Image Segmentation for testing image "Lena:" Standard Maximum Extropy

5.6. *Other Hybird Methods*

In 2006 [11], we proposed the concept of λ-measure for the best description of the connectivity or connectedness of an image. We have used the golden-cut technique for bone density measurement. It is still not very clear what value can represent the connectedness best for an image. However, it is true that we shall have such a value. So far, we have proposed at least three methods for obtaining λ values: (1) the golden-cut technique, (2) the maximum entropy method to determine the λ value, and (3) the minimum variance method for the λ value.

As we observed that λ-connected segmentation is basically a dual segmentation technique comparing to the (multiple-) threshold segmentation method that translates a gray-scale image into a binary image based on a clip-level. One can think about the threshold as a vertical processing technique, but λ-connected search is a horizontal search. These two methods share the following properties: (1) they

Fig. 9. Image Segmentation for testing image "Lena:" λ=0.98

are simple to implement, and (2) The segmentation results generate equivalence relations.

They can be cooperated by (a) using λ-connected segmentation first (to generate major segments) then multiple-thresholds (to generate detailed segments), or (b) using multiple-thresholds first (to generate major segments) then λ-connected segmentation (to generate detailed segments).

To use λ-connected segmentation first will save time in finding the appropriate thresholds. To apply thresholding first will help to stop the extreme linking.

For a general image, the best description of the image connectivity, may still be λ-measure, might only be determined in a range, however, how to find a small range to host it is the problem. Can we use information theory or statistical methods to this problem? How? It may also relate to the research on the image pyramid. [22] We need to further investigate the problems in greater detail in multiple angles.

6. λ-connectedness and Data Fitting and Reconstruction

Three most important geometric domain types of shapes are rectangles, circles, and triangles. The standard fitting methods including B-spline, Coones surfaces and Bezier polynomials are based on the rectangle shape. Even through that one can partition a circumstance into four pieces, and stretch the arc to a straight line. But it is still not a natural way to do it. To tread a triangle, one could make a fake line and split a point into two however, this will make even more odd. Therefore, the finite element method is a better way to fit a surface. The problem with the finite element method is that the computational costs are much high, especially we need to define base function for each of the triangles in the domain. These are why *lambda*-connected fitting may be necessary for some applications.

Data fitting means to fill data points based on a set of guarding sample data. Interpolation and approximation are two types of fitting methods [39]. If all of the original points remain the same value after fitting, it is called interpolation; otherwise, it is called approximation. In mathematics, there are many fitting algorithms such as B-spline, Bezier polynomial, etc. When the entire data set for a region was obtained, one can modify some points which have unusual values compared to their neighbors. This is called smoothing. Using convolution filters to eliminate high frequency points is a typical smoothing method [22]. However such a method may damage the original information.

Segmentation and fitting are two procedures that are opposites of each other. Segmentation finds the (connected) component in which each element has the same "property," so each component can be represented by an element in the component. Fittings find the (whole) distribution function based on given "representative" (sample) points. In the fitting process, if there is no sample point which was picked for a component, then the component will not appear after the fitting.

6.1. *λ-connected Fitting: Basic Algorithms*

Generally, a λ-connected fitting can be described as follows [13]: Given an undirected graph $G = (V, E)$ or a directed graph $G = (V, A)$ and a subset J of V. If $\rho_J : J \to R^m$ is known, λ-connected fitting is to find an extension/approximation ρ of ρ_J such that $\rho : V \to R^m$ is λ-connected on G (meaning that $< G, \rho >$ has only one λ-connected component) for a certain λ.

With the concept of λ-connected fitting defined, the first question to ask is: whether or not the λ-connected fitting exists and how to determine the λ-connected fitting function?

We will discuss more about the existence in next subsection. Using the same strategy as the proof of the theorem on gradually varied fill given in [3], we arrive at a variation of Theorem 2:

Theorem 2' Let G be a simple undirected graph or an acyclic directed graph. Assume that μ is given by (9) or (10). The necessary and sufficient condition under which there exists a λ-connected interpolation is that for any two vertices x and y in J , every shortest path between x and y in G is λ-connectable.

In the theorem, λ-connectable for a path π means that there is a valuation to each vertex a, $a \in G - J$, on the path such that $\beta_\rho(\pi)$ is not less than λ.

6.1.1. *Gradually Varied Fitting*

In order to make λ-connectedness easier to deal with, the concept of gradual variation instead of λ-connectedness was introduced [3]. Let $A_1, A_2, ..., A_m$ be m rational numbers, and $A_1 < A_2 < ... < A_m$. Assume f is a function from V to $\{A_1, A_2, ..., A_m\}$. If p, q are two adjacent points in Σ_2, we say f is gradually varied on p and q if $f(p) = A_i$ implies that $f(q)$ is A_{i-1}, A_i, or A_{i+1}. f is called gradually varied if f is gradually varied on any pair of adjacent points p, q in G. If $f(p) = A_i$ and $f(q) = A_j$, the level-difference between p and q is $|i - j|$. As explained in the previous section, gradual variation is a kind of λ-connection.

Let D be a connected subset in V and $J \subset D$. If given $f_J : J \to \{A_1, A_2, ..., A_m\}$, is there an extension of f_J, $f_D : D \to \{A_1, A_2, ..., A_m\}$ such that for all $p \in J$, $f_J(p) = f_D(p)$? This is called the gradually varied interpolation problem. The existence theorem is stated below:

Theorem 3 [3,7] The necessary and sufficient condition under which there exists a gradually varied interpolation is that for any two points p and q in J , the length of the shortest path between p and q in D is not less than the level-difference between p and q.

An $O(n^2)$ time algorithm for gradually varied interpolation can be found in [6,7]. The special considerations such as Jordan domains can be found in References [5]. on this Let $G = \Sigma_m$ be a discrete space which contains all integer points in the m-dimensional Euclidean space. Σ_m is also called a m-dimensional array or grid space. More generally, we can prove the constructive theorem [7]: Let i_Σ_m be an indirectly adjacent grid space. Given a subset J of D and a mapping $f_J : J \to i\_\Sigma_m$, if the distance of any two points p and q in J is not less than the distance of $f_J(p)$ and $f_J(q)$ in i_Σ_m, then there exists an extension mapping f of f_J, such that the distance of any two points p and q in D is not less than the distance of $f(p)$ and $f(q)$ in i_Σ_m. f is called a gradually varied surface. In other words, any digital manifold (graph) can normally immerse an arbitrary i_Σ_m. We can also show that any digital manifold can normally immerse an arbitrary tree T. An envelop theorem, a uniqueness theorem, and an extension theorem preserving the norm were also presented in [7] An example of gradually varied fitting is shown in Fig. 10. There are four guiding points with integer values that satisfies the condition in Theorem 2 in Fig. 10(a). The fitted ρ that is gradually varied is shown in Fig. 10(b).

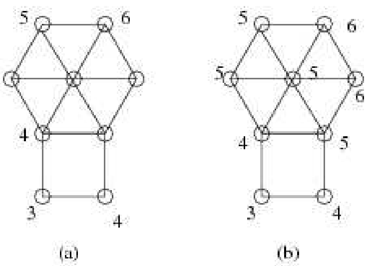

Fig. 10. Example of gradually varied fitting : (a) original graph with guiding points, (b) fitted potential function on original graph.

Traditional fitting methods such as B-splines cannot normally deal with an arbitrary graph as the domain set. Gradually varied fitting takes the advantage in such a case. However, its disadvantage is that it cannot easily get a smooth fitting. We will discuss more related issues in section 6.3.

6.1.2. *Intelligent Data Fitting and λ-connectedness*

In some situations, if there are outlier points in the sampled data, traditional fitting methods may generate faulty results. In such a case, an advanced technique called robust fitting (or regression) can be used if we assume that the data come from the same class. If the data are from different classes, such as one data point is sampled in a sandstone layer and another is sampled from a coal layer, fitting will damage the original information. In addition, in geophysics data processing, if we want to reconstruct a set of data between two points which have the same depth, when a fault exists between these two points, fitting will erase the fault. For this reason a more intelligent data fitting technique is needed.

Chen, Cooley and Zhang described an intelligent data fitting method to reconstruct a velocity volume. [?] The key idea is to perform a λ-connected segmentation first, then fit the data.

It is not very hard to implement λ-connected segmentation and to find its applications. λ-connected fitting such as gradually varied fitting has its own theorems, but it is more theoretical in nature. The discovery of more theoretical results for λ-connected segmentation and the search for more appropriate applications of λ-connected fitting are under investigation.

6.2. *Three Mathematical issues in λ-connected fitting*

There are some mathematical issues regarding λ-connected fitting: (1) λ-connected approximation if there is no λ-connected interpolation with respect to guiding points, (2) optimum fitting, and (3) λ value determination.

First, the λ-connected interpolation or gradually varied interpolation may not always exist. An approximation method can be used to get a λ-connected fitting in terms of optimal uniform approximation. [7,13] For least squared fitting, since it relates to derivatives, more tools and knowledge to deal with this problem are required.

Second, the λ-connected fitting may not be unique, so which one is the best fitting is the natural question to ask. Basically, different answers derive from different criteria. More research on this issue shall be taken place in near future.

An alternative way to solve this problem can be done by performing a λ-connected fitting first because it can handle the irregular domain easily. Then, use a standard fitting method for the secondary process. For instance, we can first generate a gradually varied surface, then use B-spline method to fit secondly when a smooth surface is required.

Third, the calculation of the value of λ in either segmentation or fitting is a critical issue. For segmentation, the selection of λ depends on how many segments we want to separate. In the following section, we will discuss the maximum-connectivity spanning tree method which can be used to find the value of λ. Since the process of finding the maximum-connectivity spanning tree is time consuming, one may use a histogram or an experimental test to solve the problem.

A binary search method can be used to test the value of λ. First, test for $\lambda = 0.5$. If λ is too big then $\lambda = 0.25$ is tried. If the if λ is too small then $\lambda = 0.75$ tried and so on. Similarly, for λ-connected fitting the same method may also be used.

6.3. *λ-connected Fitting with Gradients*

Above *lambda*-connected fitting methods only deal with "continuous" surfaces or functions. This section will present a new method to differentiable surface fitting.

As we know, *lambda*-connected fitting is to fit a surface without a math formula assumption or we do not know the formula. In addition, the domain is irregular and B-splines cannot apply to it easily.

Let $J \in D$ be the set of guiding points. If D is two dimensional space, we know $f_J : J \to R$ and $df/dv : D \to R$ then an algorithm can be developed as follows:

Step 1: Checking if f_J satisfies the λ-connected interpolation condition.

Step 2: Get start at a point p in J, get its neighborhood N_p, a) find values for all points in N_p, that has the gradient value. for df/dv at point p (with or without consider $f(p)$). Or,

b) find the best combination of valuation on N_p for df/dv at point p that keeps the original value for f_J

Step 3: Select guiding point sequentially or Randomly select p and repeats.

If not all point in D has gradient values df/dv, it is also be fine to use as many as we can.

For the step 2, we can use the method of difference analysis method to get the gradient from the values of points. Search for all possible combinations is a choice of the algorithm since N_p is a small set.

Above design philosophy is partially obtained from the finite element method. When we need to get the surface for higher smoothness, say d^2f/dv^2 is known, we can define a set that contains the neighborhood of the neighborhood of a point p to decide the fitting.

The strategy can be applied in the situation of many choices of a valuation (distance is much bigger than the level difference in Gradual Variations). This method can apply to data mining when a formula is uncertain.

Fit the *lambda*-connected surface, then to find a best fitting formula is another way to figure out the data mining solution. Otherwise one need to determine the base function ϕ and fit the surface then to find the best fit formula. We have more discussions on this aspect in [13].

7. Graph Theory and Algorithms Related to λ-Connectedness

This section presents the relationship between λ-connectedness and graph theory. We explain how the basic algorithms of graph theory are translated into λ-connectedness. These algorithms are: (1) the breadth first search algorithm, (2) Dijkstra's shortest path algorithm, and (3) Kruskal's minimum spanning tree algorithm. This section also shows how $< G, \rho >$ can be viewed as a special weighted graph if only the λ-connectivity is used for a particular application. In this case, the weights can be determined in a unified way.

7.1. *Breadth First Search for λ-Connected Components*

Breadth first search is a fast search approach to get a connected component on a graph. It first starts at a vertex p and searches for all adjacent vertices, then it "inserts" all adjacent points (neighbors) into a queue. "Remove" a vertex from the queue and let it be p, then repeatedly find p's neighbors until the queue is empty. Mark all of the vertices we visited, the marked vertices form a connected component. This technique was introduced by Tarjan [21] and was first adopted in image processing by Pavlidis [36].

To modify this idea and to use it in λ-connected search, one only needs to check if two adjacent points are λ-adjacent. The modified algorithm is shown below:

Algorithm A Breadth-first-search technique for λ-connected components.

Step 1: Let p_0 be a node in $< G, \rho >$. Set
$L(p_0) \leftarrow *$ and $QUEUE \leftarrow QUEUE \cup p_0$
i.e., labeling p_0 and p_0 is sent a queue $QUEUE$.

Step 2: If QUEUE is empty, go to Step 4; otherwise,
$p_0 \leftarrow QUEUE$ (top of $QUEUE$). Then,
$L(p_0) \leftarrow 0$.

Step 3: For each p with an edge linking to p_0 , if
$L(p) \neq 0, L(p) \neq *$, and $C(p, p_0) \geq \lambda$, then
$QUEUE \leftarrow QUEUE \cup p$ and $L(p) \leftarrow *$. Then, go to Step 2.

Step 4: Stop. $S = \{p : L(p) = 0\}$ is one λ-connected part.

7.2. *Maximum Connected Path and Dijkstra's Algorithm for Shortest Path*

According to the definition of λ-connectedness, i.e. equation (14), the connectivity between two vertices can be determined by finding a specific path between them that has the maximum connectivity. Dijkstra's algorithm [21,31] for shortest paths in weighted graphs can be modified and used for to solve this problem. The drawback of Dijkstra's algorithm is that it is slow when the size of vertices is large.

Using equation (11), we first calculate all $\alpha_\rho(x, y)$ if x and y are adjacent. That is, we obtain the weights just based on the potential function ρ. Thus, we translated $< G, \rho >$ to a weighted graph G_α. In such a situation, the neighbor-connectivity can be used to find the maximum connectivity between any two points. Based on this interpretation, one can deduce a question: How do you find a path between two nodes with the maximum connectivity. The following algorithm will generate an answer for this question.

Algorithm B The modified Dijkstra's algorithm can be used to find the connectivity from a source point to all other points in the graph [21].

Step 1:Let $T = V$. Choose the source point a
$L(a) = 1$; $L(x) = -\infty$ for all $x \in T - \{a\}$.
Step 2: Find a vertex v with the maximum $L(v)$ in T.
$T \leftarrow T - \{v\}$
Step 3: For each x adjacent to v do
$L(x) = max\{L(x), min\{L(v), \alpha_\rho(v, x)\}\}$
Step 4: Repeat step 2-3 until T is empty.

In addition, this algorithm can be associated with Dijkstra's original algorithm to solve some practical problems in transportation. Suppose that we consider both road-quality and distance. We then have two types of weights. It is appropriate to use λ- connectedness to represent road-quality, and we keep the distances as secondary weights. Therefore, we can deal with two problems: finding the shortest path under the condition of the connectivity which is not less than a certain λ, or finding the maximum connectivity path while considering the shortest path.

The following example explains how the shortest path works with the connectivity. A large retail store headquartered at Phoenix, Arizonia has two agencies in Salt Lake City, the capital of Utah, and Vernal, a small city in eastern Utah. The store wants to send the supplies to these two cities in Utah. See Fig. 11. The "potential" value of the cities is also marked on this figure.

It is assumed that the best quality roads are those of Interstate freeways. The second choice is the state highways. A big truck can be sent in the high connectivity path. For the low connectivity path, it is appropriate to send a small car. Meanwhile, the shortest roads to both Utah cities are to go through Kanab, but the highest connectivity road is to go through Las Vegas. The management will choose the path Phoenix → Las Vagas → Salt Lake City to send the supplies to Salt Lake city, and choose the path Phoenix → Kanab → Vernal for the task to Vernal. The reason for this is that since Salt lake city is a big city they must use a big truck to transfer the supplies. On the other hand, to complete the task to Vernal the management only need to sent a small car.

7.3. *Maximum Connectivity Spanning Tree and λ-Value*

The value of λ determines the number of partitions in vertex set of $< G, \rho >$. In Section 5, we addressed the problem on how do we get the value of λ based on the number of λ-connected components that was desired. In this section, the maximum connectivity spanning tree is introduced to provide the total information for what value of λ should be selected.

A spanning tree of a graph G is a tree which contains all vertices of G. A graph G has a spanning tree if and only if G is connected. A famous problem in graph theory is to find a minimal spanning tree (with the minimal total weights) for a weighted graph. The maximum connectivity spanning tree is the one in which there is a path in the tree that has the maximum connectivity for every pair of points.

Kruskal's algorithm can be used to find such a tree. [21] The tree T initially

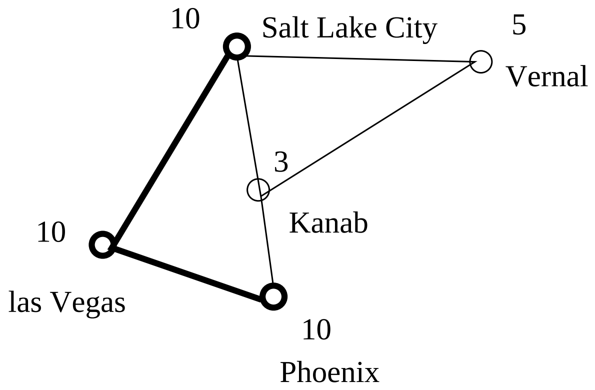

Fig. 11. The maximum connectivity path and the shortest path

contains all vertices but no edges. It then starts an iterative process: to add an edge to T under the condition that the edge has the minimum weight; however, it does not complete a cycle in T. When T has $|G|-1$ edges, the process stops.

In order to find the Maximum Connectivity Spanning Tree, it is necessary to first calculate all neighbor-connectivities for each adjacent pair in $< G, \rho >$ to form a weighted graph.

Algorithm C Modified Kruskal's algorithm can be used to find a Maximum Connectivity Spanning Tree, where $G = (V, E)$ is the original graph.

Step 1:Let $T = V$.

Step 2: Repeat step 3-4 until T has $|V| - 1$ edges.

Step 3: Find an edge e with the maximum connectivity value.

Step 4: If $T \cup e$ has no a cycle, $T \leftarrow T \cup e$ and
delete e from G; otherwise, delete e from G. Go to Step 3.

One advantage of using the maximum connectivity spanning tree is that it gives the complete information for the λ-connectivity. With this tree, one can easily get a refinement of λ-connected classification. One can also find which value of λ should be selected for a particular segmentation. For example, the number of segments that is desired is controllable. However, a disadvantage is that Algorithm C takes $O(n^3)$ time in terms of computational complexity and an extra space to store the tree. It is very slow when a large set of points/edges is considered.

After the maximum connectivity spanning tree is generated, one can easily find the all connected components for each λ. Consider an example shown below. Suppose that one wants to find out the general relationship among major cities across the United States based on size, population, and political and economic importance. We assume that the graph and the potential values of cities are shown in Fig. 12.

For simplicity, formula (16) is used to compute the neighbor-connectivity, i.e. $\mu = 1 - |u - v|/10$. Then, the connectedness on each edge can be calculated. The result is shown in Fig. 13.

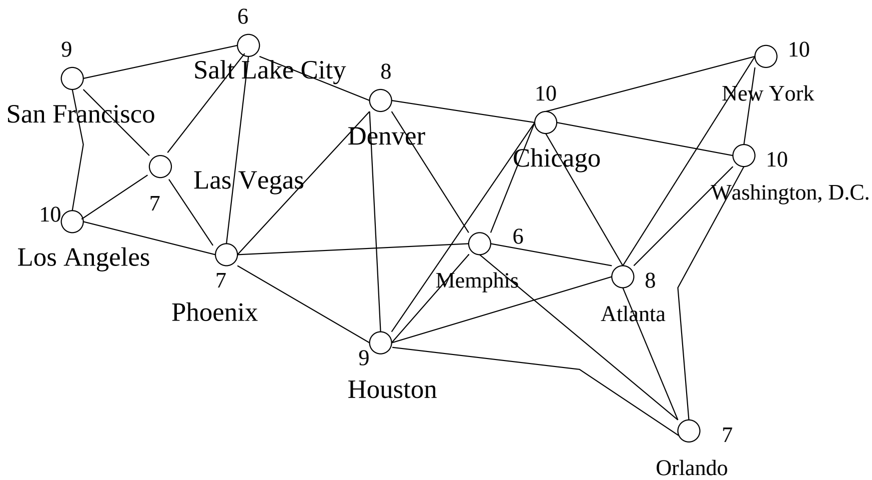

Fig. 12. Example of the potentials for the major cities in US

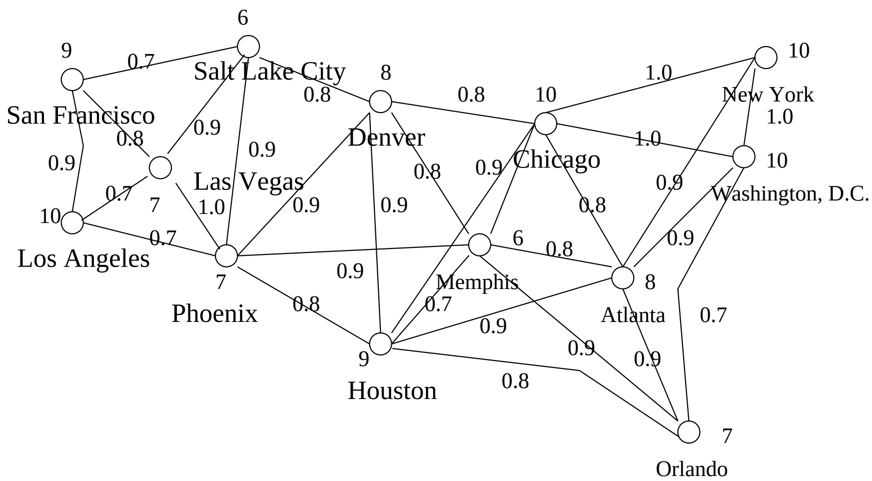

Fig. 13. The connectivity map based on the potential function.

According to Algorithm C, the maximum connectivity spanning tree can be computed and shown in Fig. 14. When λ is set to be 0.8 or smaller, it is always to generate the same λ-connected component(s) that contains all cities. If λ is 0.9, then there are two λ-connected components: one contains San Francisco and Los Angeles, and another one contains the rest of cities. Therefore, a maximum connectivity spanning tree provides full information of the λ-connectedness among vertices.

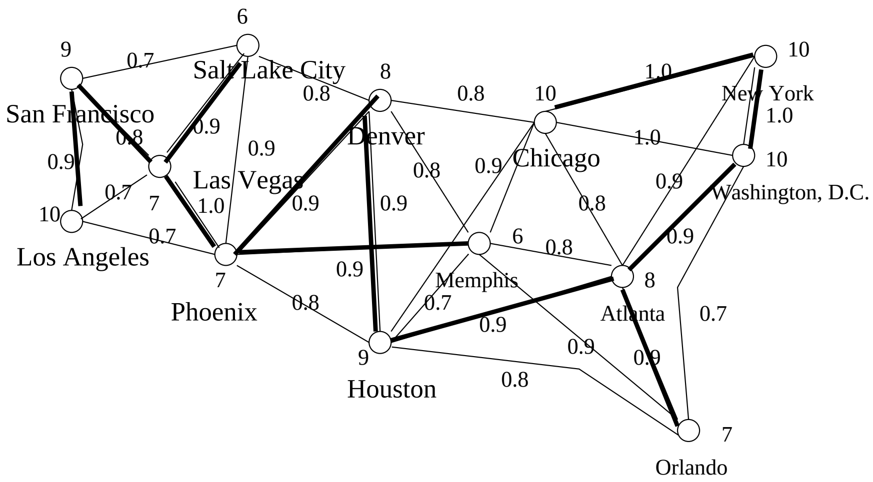

Fig. 14. The maximum connectivity spanning tree for Fig. 13.

8. Applications Related to λ-connectedness

Although we have already discussed some of the applications related to λ-connectedness, in this section we will attempt to enlarge our scope and link it with state of the art methods in artificial intelligence: curve tracking in target recognition, rough sets theory, Baysian networks, and network economics. To use λ-connectedness in applications, we first needs to map a problem to a graph G. Since solving this problem needs to measure the relationship of any pair of adjacent nodes, we need to obtain the potential function over the notes ρ. Then, λ-connectedness on the $< G, \rho >$ can be determined.

8.1. *λ-connected Curve Search*

An arbitrary shaped curve tracking is used in many applications. A typical example is the tracking of an aircraft in space based on radar images. This problem is mathematically equivalent to extracting a curve from an image. It becomes more complex when more objects appear in the same target area. Previously, techniques have been developed for finding such lines or circles in an image. [22]

Curve tracking usually deals with binary images; however, occasionally, one needs to deal with a gray-scale image. For example, in order to reduce noise, we can translate an image containing only "dots" into a gray-scale image in ionogram scaling. Assume an image contains only dots and line/curve-looking-segments. The task is to extract one or more curves formed by dots and line-segments.

A technique was designed to track these curves by Chen, Berkey and Johnson in. [15] The technique includes three major steps: (1) Using λ-connected segmentation (searching), determine all connected components in the image. The adjacency (search) region of a point can vary. (2) Search all possible sub-curves in a (λ-connected) component. (3) Use a genetic algorithm to link sub-curves and to track the best curves. This method was used to extract E-layer and F-layer curves in ionograms. Tsai, Berkey and Liu [46] designed a automatic scaling system using λ-connected search and fuzzified linking. They proposed three measure functions μ_1, μ_2, μ_3 for amplitude, frequency, and range connectedness, respectively.

$$\mu_1(u,v) = 1 - \frac{\min(\Delta Amp(u,v), \Delta Amp_{max})}{\Delta Amp_{max}}, \tag{24}$$

$$\mu_2(u,v) = \frac{\Delta F_{max} - \max(\Delta F(u,v) - \Delta F_{min})}{\Delta F_{max} - \Delta F_{min}}, \tag{25}$$

$$\mu_3(u,v) = \frac{\Delta R_{max} - \max(\Delta R(u,v) - \Delta R_{min})}{\Delta R_{max} - \Delta R_{min}}. \tag{26}$$

Previous research and results related to this application can be found in [45].

8.2. λ-connectedness and Rough Sets

Rough sets were proposed by Pawlak in 1982. [37] It was used to describe an unknown set by using its absolute true subset called "lower approximation" and its absolute true containing set called "upper approximation." Rough sets find their use in data mining. Chen discussed a limited multi-level λ-connected search accomplished with rough sets in data processing. [9]

This section attempts to use λ-connectedness to express rough sets and to determine the "lower approximation" and the "upper approximation" by finding two special λ-connected sets. It possibly allows us to apply all algorithms discussed in section 7 to rough sets and related data mining problems.

Mathematically, given an equivalence relation R on the base set U, U can be partitioned by R into disjoint components, $P_1, ..., P_k$. Let X be a subset of U (See Fig. 15 (a)). The lower approximation $L_R(X)$ and upper approximation $U_R(X)$ are defined below:

$L_R(X) = \cup_j \{P_j | P_j \subset X\}$

and

$U_R(X) = \cup_j \{P_j | P_j \cap X \neq \emptyset\}$.

In other words, $L_R(X)$ is the union of interior equivalence classes of X (See Fig. 15 (b)) and $U_R(X)$ is the closure of X with respect to R (See Fig. 15 (c)). Thus,

$$L_R(X) \subset X \subset U_R(X). \tag{27}$$

By the way, to use rough sets in data mining, it is necessary to first define an equivalence relation R on the base set U based on some incomplete knowledge. From the above expression, it can be deduced that one only needs to deal with the set of $U_R(X) - L_R(X)$. If another piece of knowledge R' for U is found, one could apply the same reasoning to $X' = U_R(X) - L_R(X)$ and only needs to deal with $U_{R'}(X') - L_{R'}(X')$. This is an iterative process that could save considerable computational resources such as time and memory.

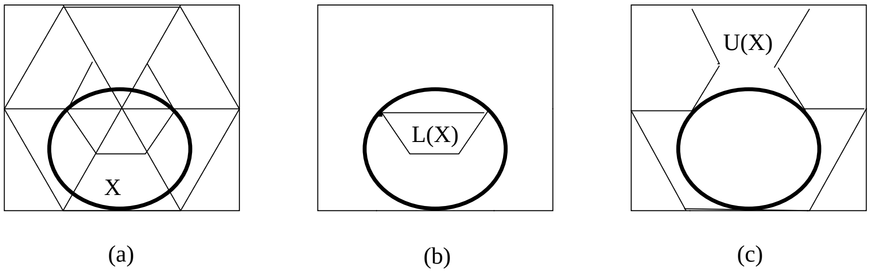

Fig. 15. A rough set example: (a) Original partition and a set X, (b) $L(X)$, the "lower approximation" of X, and (c) $U(X)$, the "upper approximation" of X.

8.2.1. *λ-connected Representation for Rough Sets*

Assume that U is the base set (or the universal set) and R is an equivalence relation. Let R partition U into k components P_1,...,P_k, and an arbitrary set $X \subset U$. We need to create a graph $G = (V, E)$ and a potential function ρ.

Let $V = \{P_1, ..., P_k\}$, and $E = \{(P_i, P_j)|i \neq j; i, j = 1, ..., k\}$. and $f(P_i) = \frac{|P_i \cap X|}{|P_i|}$.

is called the rough membership function [37]. Thus, $L_R(X) = \cup\{P_i|f(P_i) = 1\}$, and $U_R(X) = \cup\{P_i|f(P_i) > 0\}$. Since G is a complete graph in this case, for any kind of C_ρ, $L_R(X)$ can be searched by assigning $\lambda = 1$ and starting at a P_i with $f(P_i) = 1$. $U_R(X)$ can be searched by letting $\lambda > 0$ and starting at P_i with $f(P_i) > 0$. Furthermore, λ-connected search can also acquire more detailed information by assigning different values of λ. In other words, λ-connectedness not only can represent a rough set, but also get a complete hierarchy information for the rough set with respect to the value of λ.

8.2.2. *Normal λ-connected Sets and Rough Sets*

In Section 8.2, we mentioned normal λ-connected sets. A normal λ-connected set is that every pair of adjacent nodes (with its potential function values) are λ-adjacent. So, it is a subset of the general λ-connected component. Normal λ-connected sets are a strict representation while the general λ-connected component is a loss representation. This is philosophically similar to rough set theory. For λ-connectedness, any subset of U, X, can be represented by the "lower" segmentation containing X (approximation) which is the union of all normal λ-connected components each of which has a point in X, and "upper" segmentation (approximation) which is the union of all λ-connected components each of which has a point in X. When U is a symbolic space, λ-connectedness is still applicable depending on the specific situation.

8.3. λ-connectedness and Belief Networks

Belief Networks are also called Baysian networks that are used for inference and reasoning. Let $U = \{X_1, ..., X_n\}$ be a set of random variables. A directed graph is used to represent the these variables (as vertices) and the causal connection (as arcs/edges). For example, if there is an arc from vertex X to vertex Y, we know that X causes Y (see Fig. 16). Each node associates with a table that indicates the conditional probability, called conditional probability (distribution) table. [25, 34] The summation of all outer probabilities is equivalent to the probability on the node. The solving path is to backpropagate the source. This is the reason why a neural network can be viewed as a type of Baysian networks.

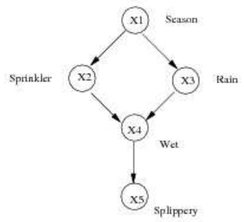

Fig. 16. Example for Baysian networks

For the example shown in Fig. 16, if the grass is slippery, then it is wet. This is either caused by rain or the sprinkler which was turned on. The Baysian network can tell us which one is most likely based on the joint distribution formula.

$P(x_1, ..., x_n) = \Pi_i P(x_i|pa(i))$

where x_i is the value of X_i, and $pa(i)$ is all values of parent notes of X_i.

Therefore, the weight of each arc will become reality after an assumption is made or a real event that has appeared. The path of propagation reasoning is then similar to a λ-connected path. When the majority factor is closed to the secondary factor, it is very difficult to distinguish which one is the true cause. Using λ-connected search to find a connected component is very helpful in directed graph. [31]

Both Baysian networks and λ-connectedness method are based on graphs, however, they are facing different tasks. The solution path in Baysian networks should have the maximum λ-connectivity when an event occurred. A Baysian network is intended to find a source whereas λ-connected search is to find "an end." Philosophically, λ-connected search starts at the source then to find the all possible connections. Baysian networks start at the result then to find the source. For a complicated system, if one can partition the network into "connected components," then the calculation in Baysian networks will be much easier.

In addition to the relationship between λ-connectedness and random graphs [1], we know that in a random graph, the "weights" assigned to each edge is the same probability value. Thus, it is a very accurate method even through it uses probability. However, it is hard to find many applications of computer science that have evenly distributed probability on each edge. We can summary the similarities and the differences among these three concepts in Table 1.

	Foundation	Accuracy	Inform.required	Speed	Data size
BN	Graph, Prob	High	Complete	Depends	Depends
λ-CM	Graph, Fuzzy	Depends	flexible	Fast	Can be Large
RG	Graph, Prob	High	Limited	Fast	Can be Large

8.4. *λ-connectedness: The Potential Applications in Network Economics*

This section discusses some potential uses of λ-connectedness in resource management and network economics. Economics is the study of how people choose to use their scarce and limited resources in an attempt to satisfy their unlimited wants. In resource markets, when a demand matches a supply, then the situation meets an equilibrium. [23, 33] Network economics apply graph theory and computational economics method to study equilibrium problems. [23]

Consider an example of the structure of a human resource network. In modern times, the structure may not appear to be a hierarchical tree structure but a network structure. Each node p represents a person, the potential function $\rho(p)$ represents his/her wage/duty/power.

Assume a company wants to re-organize its structure. They want to add a new node, a new position. It is required to know how much power the institution would give to the person who will take this position. This becomes an interpolation problem for ρ. Let's suppose that there is no "gap" in the potential function ρ after adding this node. This is also the basis that the employer uses to find a person who will best fit the position. In such a case, a gradually varied fitting method can be used to find a solution. If the wage/duty/power function is an m-dimensional vector, each component of the vector must be fitted individually. In the case of a directed graph, we may require that the graph is acyclic.

The second example is as follows: Suppose that a computer corporation would like to build a research center in Salt Lake City (SLC). The center will need to hire researchers to work on computer complexity and quantum computing, but the wages for these positions are limited. There are only a few universities which "produce" Ph.D. s in such fields. Suppose that these universities are HU at Boston, PU at Princeton, and UC at Berkeley. There are several major cities between SLC-Boston and SLC-Princeton, such as Chicago and Denver. Most new Ph.D. s from HU and PU will likely want to go Chicago or Denver but not SLC. Because both Denver and

Chicago are more "attractive" for those from HU and PU. The "attractiveness" of a city can be the potential function for λ-connectedness. Because the "attractiveness" varies with the distance, the potential function will have three components $\rho = (f_{Bos}, f_{Pnt}, f_{Bky})$. Suppose it is known that the potential values on the vertices of Chicago and Denver. We want to interpolate for SLC which the company requires. It is assumed that the path $\{Bos, Chi, Den, SLC\}$ is a λ-connected path with respect to new Ph.D. s in computational complexity and quantum computing. The calculation result will show whether or not the company will be able to hire the Ph.D. s from these universities.

9. Summary and Discussion

The λ-connectedness approach was designed for real image processing problems such as segmentation, searching, and data reconstruction. This paper has utilized them into a large scope by using graph-theoretic methods. The advantages of the integration are: 1) a sound background, 2) more application areas such as management networks, rough sets, and network economics, and 3) many existing fast algorithms in graph theory that can be directly used for λ-connectedness, such as the breadth-first search algorithm, Dijkstra's Algorithm, and Kruskal's algorithm. λ-connectedness also has relationships with functional analysis and topology that we shall investigate in depth in the future. [10]

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