8. Gradually Varied Functions on Digital and Discrete Manifolds

The gradually varied surface was introduced and studied in digital and discrete spaces by Chen in 1989 [6]. The basic idea of introducing gradually varied surfaces is to employ a purely discrete interpolation algorithm to fit a discrete surface when the desired surface is not required to be "smooth."

In this Chapter, we will introduce the generalized concept of gradually varied surfaces. A digital surface is defined as a mapping f from an n-dimensional digital manifold D into an m-dimensional grid space Σ_m. A discrete surface is said to be gradually varied if two points p, and q are adjacent in D, implying that $f(p)$ and $f(q)$ are adjacent in Σ_m.

We will present the following constructive theorem: Let i_Σ_m be an indirectly adjacent grid space. Given a subset J of D and a mapping $f_J : J \to i\_\Sigma_m$, if the distance of any two points p and q in J is not less than the distance of $f_J(p)$ and $f_J(q)$ in i_Σ_m, then there exists an extension mapping f of f_J, such that the distance of any two points p and q in D is not less than the distance of $f(p)$ and $f(q)$ in i_Σ_m. That is to say, the guiding point set $(J, f(J))$ has a gradually varied surface fitting. In other words, any digital manifold (graph) can normally immerse an arbitrary i_Σ_m. We also show that any digital manifold (graph) can normally immerse into an arbitrary tree T.

Furthermore, we will discuss the gradually varied function, which is a gradually varied surface in the case of $m = 1$ in i_Σ_m or integer set Z. An envelope theorem, a uniqueness theorem, and an extension theorem which concerns preserving the same norm, are obtained. Finally, we will show an optimal uniform approximation theorem of gradually varied functions and develop an efficient algorithm for the approximation.

8.1 Background

Before Newton-Leibnitz's time, mathematics was basically "discrete." Since then, continuous mathematics has dominated the literature. However, discrete mathematics has found new life with the appearance and widespread use of the digital computer. However, we still prefer to use the thinking involved in continuous mathematics. For example, if we had discrete information on some samples, we would assume a continuous model to do the calculation.

Sometimes, we need a discrete output from the continuous solution, and it is not hard to re-digitize the continuous results.

For some problems, going from "discrete" to "continuous" back to "discrete" may not always be necessary. In such instances, we can directly employ a methodology to go from "discrete input" to "discrete output." The tractability and practice of the methodology using such a philosophy is certainly valid. Let's consider an example. In seismic data processing, the seismic data sets consist of synchronous records of reflected seismic signals registered by a large number of geophones (seismic sensors) placed along a straight line or in the nodes of a rectangular lattice on the earth's surface. A series of explosions serve as the source of the initial seismic pulse, responses to which are averaged in a special manner. The vertical time axis forming the resulting two- or three-dimensional picture is identified with depth, so that the peculiarities of the reflected signal under the respective sensor carries information on the local properties of the rock mass at the respective point of the underground medium. In contrast to the above-lying sedimentary cover, the absence of pronounced reflecting surfaces in a crystalline body makes it difficult to infer the geological information from the basement interval of the seismic picture. We can see that layer description (or modeling) becomes a central problem. If we know a target layer in each horizontal or vertical (line) profile, we can get the entire layer in the 3D stratum. It can be transferred into a surface fitting problem where we can use a Coons surface, Bezier polynomial, or B-spline to fit the surfaces. See Fig. 8.1.

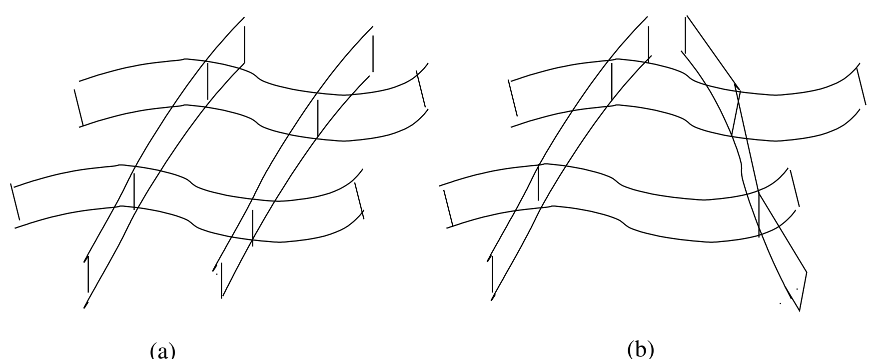

Fig. 8.1. (a)The layer described by perpendicular lines and (b)the layer described by arbitrary lines.

Based on the boundary values to fill the interior, the most suitable technique is a Coons surface. However, for a layer, one must make two surfaces, one for the top of the layer and one for the bottom. The Coons surfaces have no property of preserving a fitted surface in the convex of a guiding point set. That is to say, the upper surface may intersect with the lower surface.

That is not a desired solution. Since there are many sampled points on measured lines, the Bezier polynomial is also not a good choice–One cannot make the order of the polynomial very high. B-spline is a very good choice for the problem, but we need to do a pre-partition and coordinate transformations. In fact, for the problem, we have no special requirements for the smoothness, and we just need two reasonable surfaces to cover the layer.

Another example is from computer vision. In observing an image, if you extract an object from the image, a representation of the object can sometimes be described by its boundary curve. If all values on the boundary are the same, then we can just fill the region. If the values on the boundary are not the same, and if we assume that the values are "continuous" on the boundary, then one needs a fitting algorithm to find a surface. If the boundary is irregular, one needs to use a 2D B-spline to divide the boundary into four segments. The different partitions yield different results.

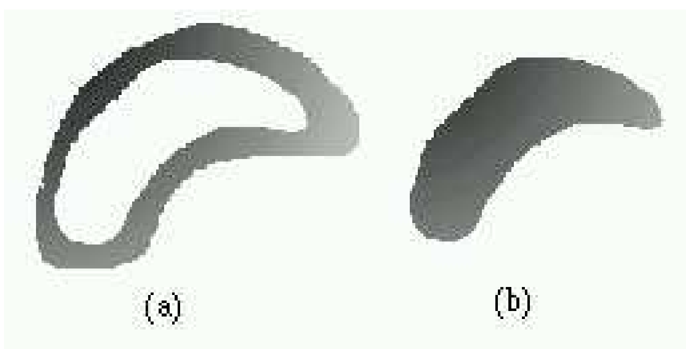

Fig. 8.2. An irregular gradually varied boundary.

Practically, the procedure of a computation is a set of "discrete" actions. The input of a curve is discrete, and the output is also discrete. We can, therefore, make the following arguments. Do we always need a continuous technique for surface fitting? Is it possible to have a discrete fitting algorithm to get a reasonable surface for the above problems? In 1989, Chen developed an algorithm to do such discrete surface fitting in 2D [6]. The algorithm is called gradually varied fitting.

This chapter will expand on and generalize the idea of discrete surface fitting on a plane to a generalized discrete space, namely a discrete manifold. We will consider building a systemic theory over the discrete manifold. We will first describe a discrete manifold, then define the gradually varied surfaces in the discrete manifold. Then, we will give the major theorems of this chapter, which are the theorem of interpolation and the expanding theorem of gradually varied surfaces. We will also discuss the gradually varied manifold over discrete manifolds.

We will go on to consider several kinds of gradually varied operators and study the properties of these operators. We will also define the gradually varied functional and its norm. We will prove the expanding theorem that preserves the norm of a gradually varied function. We will also prove the uniqueness theorem and its corollary. We will also discuss the approximation problem of gradually varied functions.

Since the purpose of introducing gradually varied surfaces is for applications in image processing, the existence of fast algorithms is a really significant issue. We will develop fast algorithms for the interpolation and approximation of gradually varied surfaces. We will also discuss the efficient filling algorithms for the discrete Jordan manifolds.

8.2 Gradually Varied Mapping

The gradually varied surface was originally designed for digital surface fitting without using any assumption of continuous functions such as splines and Bezier polynomials in geophysical data processing [6][7][9]. A gradually varied surface can be viewed as a discrete function f from Σ_2 to $\{1, 2, ..., n\}$ satisfying $|f(a) - f(b)| \leq 1$ if a and b are adjacent in Σ_2. This concept is called "discretely continuous" by Rosenfeld [68] and "roughly continuous" by Pawlak [61]. A gradually varied function can be represented by λ-connectedness [5][7][20] [22].

Definition 8.2.1. *Let D_1 and D_2 be two discrete manifolds and $f : D_1 \to D_2$ be a mapping. f is said to be an immersion from D_1 to D_2 or a gradually varied operator if x and y are adjacent in D_1 implying $f(x) = f(y)$, or $f(x)$, $f(y)$ are adjacent in D_2.*

If $D_2 = \Sigma_m$, then f is called a gradually varied surface. An immersion f is said to be an embedding if f is a one-to-one mapping.

In fact, D_1 and D_2 can be two simple graphs in the above definition. In this case, we know a famous NP-complete problem [1][31], the subgraph isomorphism problem, is related to the gradually varied operator.

Lemma 8.2.1. *There is no polynomial time algorithm to decide if a graph D can embed to another graph D' unless $P = NP$.*

However, if the number of adjacent points to each point in D and D' are not greater than a constant C and $|D| = |D'|$, then there is a polynomial time algorithm to decide if $|D|$ and $|D'|$ are isomorphic. On the other hand, the immersion problem is trivial because we can always let the image of all points in D be a certain point in D'.

There is another related NP-complete problem. An immersion is said to be a morphism if for $a' = f(a)$, where b' is an adjacent point of a', then there is a $b \in f^{-1}(b')$ such that a, b are adjacent. To decide whether or not an immersion is a morphism is NP-hard.

This chapter does not deal with the above complexity problems. It focuses on how to get the entire mapping using the property of immersion for a given part of a mapping, .

Mathematically, the main problem of the chapter: Let D and D' be two discrete manifolds. Assume J is a subset of D, if $f_J : J \rightarrow D'$ is known, then there is an extension of f_J, f_D, such that $f_D : D \rightarrow D'$ is gradually varied, where an extension means that $f_J(a) = f_D(a)$ if $a \in J$.

This problem has direct use in image reconstruction. A 2D gray-scale digital image is a mapping $f : \Sigma_2 \rightarrow [0, 1]$. Let's consider an example. In geophysical prospecting, there is an important problem, namely layer description or modeling, which uses an arrangement of nets consisting of horizontal and vertical lines. If we know the object layer in each horizontal or vertical line, we can get the entire layer in the stratum. This is a simple surface-fitting problem, where we could use a Coons surface, Bezier polynomial, or a B-spline to fit the surface. However, we can represent the complete gray-level (usually 0-255) as a chain, which links each pair of $(i-1, i)$, $i \in \{1, 255\}$. To fit the surface is to find a gradually varied function with respect to the chain. See Fig. 8.1.

It is easy to see that if f is a gradually varied mapping, then

$$\forall_{p,q \in D} d(p, q) \geq d(f(p), f(q)), \tag{8.1}$$

where $d(x, y)$ is the length of the shortest path between x and y. The key theorem for extension was proven by Chen in 1989 [6][9], which states:

Theorem 8.2.1. *Let $A_1, A_2, ..., A_m$ be m rational numbers with $A_1 < A_2 <, ..., < A_m$, and let J be a nonempty subset of graph D. Suppose that $f_J : J \rightarrow \{A_1, A_2, ..., A_m\}$, the necessary and sufficient condition for the existence of a gradually varied extension, f_D is for all x, y in J, $d(x, y) \geq d(f_J(x), f_J(y))$, where $d(f_J(x), f_J(y)) = |i - j|$ if $f_J(x) = A_i$ and $f_J(y) = A_j$.*

We also knew that [7][10],

Theorem 8.2.2. *If there is a gradually varied extension f_D of f_J, then there exists two gradually varied extensions f_D^1 and f_D^2 such that, for every gradually varied extension f_D, $f_D^1(x) \leq f_D \leq f_D^2$ for all $x \in D$.*

We also discussed fast algorithms for the gradually varied extension in [7][10].

8.3 Gradually Varied Extension and Normal Immersion

We can see that Theorem 8.2.1 is an ideal case. However, not every D, D' can preserve such a property, i.e., the existence of gradually varied extension.

For example, consider D and D' as shown in Fig. 8.3(a) and (b), respectively. If $J = \{a, b, c, d\}$ and f_J is indicated in Fig. 8.3 (b), then we can see that $< J, f_J >$ satisfies the condition in Theorem 8.2.1. But for point x, there is no $f(x)$ that can be selected to satisfy the requirement of gradual variation.

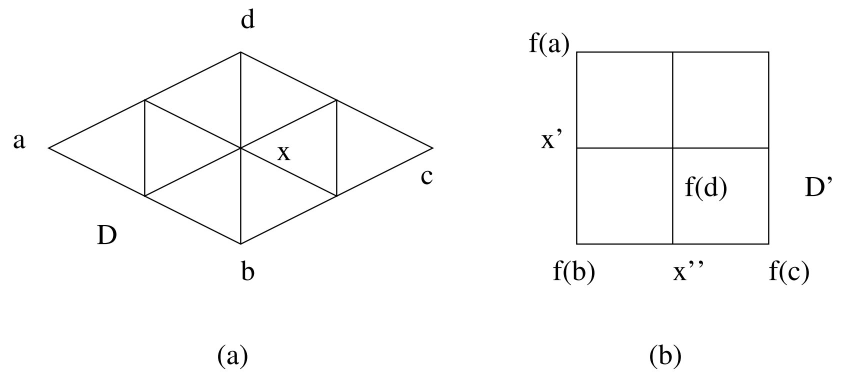

Fig. 8.3. The example that shows no gradually varied extension.

In order to do more profound research and make the terminology simpler, we propose a new concept as follows:

Definition 8.3.1. *Let J be a subset of D and f be a mapping $f_J :\rightarrow D'$, which satisfies:*

$$\forall p, q \in J[d(p, q) \geq d(f_J(p), f_J(q))]. \tag{8.2}$$

If there exists an extension f of f_J such that $f : D \rightarrow D'$ is a gradually varied mapping, then we say $< J, f_J >$ is immersion-extendable. If every $< J, f_J >$ satisfying (3.1) is immersion-extendable, then we say that D can normally immerse into D'.

Mathematically, it might be very difficult to decide whether or not a general graph/digital manifold can normally immerse into another one. This problem may be NP-hard [31]. However, in the use of data fitting, pattern recognition, and computer vision, we can find several important and useful digital manifolds in which normal immersion and general gradually varied surfaces are applicable. In other words, given some guiding points and their (mapping) values in such a discrete manifold, we can find a gradually varied surface based on the guiding points.

The m-dimensional grid space Σ_m is the set of all points in n-dimensional Euclidean space with integer coordinates. There are many kinds of adjacency, and we present two of them here, which are mostly often used in discrete geometry. First, two points $p = (p_1, ..., p_m)$ and $q = (q_1, ..., q_m)$ are directly adjacent if and only if $\sum_{i=1}^{m}(|p_i - q_i|) = 1$. Such a digital space is called a directly adjacent space and is denoted by d_Σ_m. Second, two points $p = (p_1, ..., p_m)$ and $q = (q_1, ..., q_m)$ are indirectly adjacent if and only if $Max_{i=1}^{m}(|p_i - q_i|) = 1$. Such a digital space is called an indirectly adjacent space and is denoted by i_Σ_m. In addition, the n-simplicial decomposition space is denoted by Δ_n.

We know that Theorem 8.2.1 states that any digital manifold can normally immerse to Σ_1 or a chain. We can show a more general theorem in the following:

Theorem 8.3.1. *Any graph D (or digital manifold) can normally immerse an arbitrary tree T.*

Proof Suppose that $J \neq \emptyset$ is a subgraph of D, and a mapping $f_J : J \to T$ satisfies (8.1), where T is a tree. We will prove that there exists a gradually varied mapping $f : D \to T$ with $f_J(p) = f(p)$, $p \in J$.

The proof of the theorem is a constructive proof. For every $p \in J$, let $f(p) = f_J(p)$, and let $f(p) = \theta$ if $p \in D - J$. Denote

$$D_0 = \{p | f(p) \neq \theta \& p \in D\}. \tag{8.3}$$

For any p and p' in D_0, we have $d(p, p') \geq d(f(p), f(p'))$.

If $D_0 \neq D$, we can find a point x in $D - D_0$ such that x has an adjacent $r \in D_0$, then we have two cases:

1) If $d(r, p) > d(f(r), f(p))$ for every $p \in D_0$ with $p \neq r$, then let $f(x) = f(r)$. Since $d(r, x) = 1$ is not less than $d(f(x), f(r)) = 0$ and

$$d(x, p) \geq d(r, p) - 1 \geq d(f(r), f(p)) - 1,$$

$d(x, p) \geq d(f(r), f(p)) \geq d(f(x), f(p))$. Let $D_0 \leftarrow D_0 \cup \{x\}$ and return to repeat.

2) If there is a point $p \in D_0$ such that $d(r, p) = d(f(r), f(p))$, then let

$$R_1 = \{p | d(r, p) = d(f(r), f(p)), p \in D_0\}. \tag{8.4}$$

According to (8.3), there are only two instances:

2.1) If for every p in R_1, $d(r, p) < d(r, x) + d(x, p)$, then $d(r, p) < 1 + d(x, p)$, so $d(x, p) \geq d(r, p)$. Let $f(x) = f(r)$. If $p \in R_1$, then $d(x, p) \geq d(r, p) \geq d(f(r), f(p)) \geq d(f(x), f(p))$; else, i.e., p in $D_0 - R_1$, since $d(r, p) > d(f(r), f(p))$ then

$$d(x, p) \geq d(r, p) - 1 \geq d(f(r), f(p)) \geq d(f(x), f(p)).$$

Let $D_0 \leftarrow D_0 \cup \{x\}$ and return to repeat.

2.2) If there is some p in R_1 such that

$$d(r, p) = 1 + d(x, p) = d(r, x) + d(x, p),$$

then x is obviously the shortest path between r and p. In the following, we will fix the point p. For every $q \in D_0$ with $q \neq r$ and $q \neq p$, we have $d(p, q) \leq d(p, x) + d(x, q)$. Let

$$m(x) = \{q | d(x, q) < d(q, r), q \in D_0\}. \tag{8.5}$$

There are only two cases to discuss:

2.2.1) If $q \in m(x)$, then $d(p,q) < d(p,x) + d(x,q) < d(r,p) - 1 + d(r,q)$. Also, since p belongs to R_1,

$$d(p,q) \le d(r,p) + d(r,q) - 1, \tag{8.6}$$

and

$$d(p,q) \le d(f(r), f(p)) + d(r,q) - 1. \tag{8.7}$$

Three cases can be derived from the above equation (8.6):

2.2.1.1) If $d(r,p) \ge d(f(r), f(q)) + 2$, then $d(x,q) \ge d(r,q) - 1 \ge d(f(r), f(q)) + 1$. Therefore,

$$d(f(x), f(r)) \le 1 \text{ implies } d(x,q) \ge d(f(x), f(q)).$$

2.2.1.2) If $d(r,q) = d(f(r), f(q))$, then by using (8.5) and $p \in R_1$,

$$d(p,q) < d(f(r), f(q)) + d(f(r), f(p)) - 1. \tag{8.8}$$

Because T is a tree, T has one and only one path between $f(p)$ and $f(p)$, which must be a shortest path. If this path contains the point $f(r) \in T$, then

$$d(f(p), f(q)) = d(f(r), f(q)) + d(f(r), f(p)) \tag{8.9}$$

and by using (8.7), we can obtain

$$d(p,q) \le d(f(p), f(q)) - 1.$$

A contradiction occurs for the assumption of p and q in D_0. Hence, this path is not through $f(r)$.

2.2.1.3) If $d(r,q) = d(f(r), f(q)) + 1$, then

i) If $d(x,q) \ge d(r,q)$, then $d(x,q) \ge d(f(r), f(q)) + 1$. This case is similar to 2.2.1.1).

ii) If $d(x,q) = d(r,q) - 1$ and according to the fixed p with $d(r,p) = 1 + d(x,p)$ and $d(r,p) = d(f(r), f(p))$, we have

$$d(p,q) \le d(p,x) + d(x,q) \le d(r,q) + d(r,p) - 2,$$

so

$$d(p,q) \le d(f(r), f(q)) + 1 + d(f(r), f(p)) - 2.$$

It is similar to 2.2.1.2).

2.2.2) If $q \in D_0 - m(x)$, i.e., $d(x,q) \ge d(r,q) + 1$, then

$$d(x,q) \ge d(f(r), f(q)) + 1.$$

It is similar to 2.2.1.1).

To summarize, if there is a point p in R_1, such that

$$d(r,p) = 1 + d(x,p) = d(r,x) + d(x,p),$$

then for any q in $D_0 - \{r,p\}$, it must be included in M_1 or M_2, where M_1 is

$$M_1 = \{q \mid d(x,q) \ge d(f(r), f(q)) + 1, q \in D_0 - \{r,p\}\}$$

.

We can choose $f(x) = f(r)$ or make $f(x)$ adjacent to $f(r)$ in the tree T.

$M_2 = \{q|$ the shortest path between $f(p)$ and $f(q)$ that is not through $f(r)$, $q \in D_0 - \{r,p\}$ $\}$.

Let $F_2 = \{f(q)|q \in M_2\}$. We can see that the point $f(r)$ is the root of the tree T. It is easier to see that every point in $F_2 \cup \{f(p)\}$ is included in a subtree of the tree. Thus, there exists a node t in T, such that $d(f(r),t) = 1$ and

$$\forall_{s \in F_2 \cup \{f(p)\}} d(f(r),s) = d(t,s) + 1.$$

Thus, let $f(x) = t$. In fact, if the path between $f(r)$ and $f(p)$ is $f(r)$, s_1 ,..., s_{l-1}, $f(p)$, then $t = s_1$. Thus, the proof of Theorem 8.3.1 is completed. ♢

Corollary 8.3.1. *Any graph/digital manifold can normally immerse into an arbitrary forest.*

Theorem 8.3.2. *Any graph/digital manifold D can normally immerse into i_Σ_m.*

Proof: Let D be a digital manifold. Suppose that J is a subset of D and $J \neq \emptyset$, and a mapping $f_J : J \to i\_\Sigma_m$ satisfies (8.1). We can denote $f_J(x) = (f_J^{(1)}(x), f_J^{(2)}(x), ..., f_J^{(m)}(x))$, and for any two points p and q, we have,

$$d(p,q) \geq d(f_J(p), f_J(q)).$$

Since

$$d(f_J(p), f_J(q)) = Max\{d(f_J^{(k)}(p), f_J^{(k)}(q)) | k = 1, ..., m\},$$

then

$$d(p,q) \geq d(f_J^{(k)}(p), f_J^{(k)}(q)), k = 1, ..., m.$$

Let $f_J^{(k)} : J \to i\_\Sigma_m$, where

$$f_J^{(k)}(x) = (0, ..., 0, f_J^{(k)}(x), 0, ..., 0). \qquad (8.10)$$

By theorem 3.1, we have the total of m gradually varied extensions as follows:

$$f^{(k)} : D \to i\_\Sigma_m, k = 1, ..., m. \qquad (8.11)$$

Finally, we can combine these m gradually varied extensions into a unique mapping:

$$f(x) = \sum_{k=1}^{m} f^{(k)}(x).$$

It preserves $f(p) = f_J(p)$, $p \in$J. Also, if p and q are adjacent in D, then by (8.10)

$$d(f(p), f(q)) = Max\{d(f[k](p), f[k](q)) | k = 1, ..., m\} \leq 1.$$

Hence, f is a gradually varied extension of f_J. ♢

Theorem 8.3.3. *(1) A d_Σ_m cannot normally immerse into itself, for $m >$* 1.
(2) A Δ_n, $n > 2$, cannot normally immerse into itself.
(3)A d_Σ_m or i_Σ_m cannot normally immerse into Δ_n, for $n > 2$. (4) A Δ_n, $n > 2$, cannot normally immerse into d_Σ_m.

Proof: We only need to list some counter-examples to prove this theorem.

(1) Assume the D, D' which are given in Fig. 8.4 (a), (b). If $J = \{p, q, s, r\}$ and $f_J = \{p', q', s', r'\}$ with $f_J(a) = a'$, we know for all $a, b \in J$, $d(a, b) \geq d(f_J(a), f_J(b))$. However, for $x \in D$, $f_J(x)$ can only be assigned as x', or x''. Neither x' nor x'' can satisfy the condition of gradual variation.

(2) For $< \Delta_n, \Delta_m >$, $m, n > 2$, let the mapping f_J have the domain $J = \{a, b, c\}$ and the range $f_J = \{a', b', c'\}$ in Fig 8.4 (c), (d). We cannot find the image of $x \in D$ satisfying the condition of gradual variation.

(3) For $< d\_\Sigma_m, \Delta_n >$, the proof is similar to (2) with respect to Fig. 8.4 (e), (f).

(4) For $< i\_\Sigma_m, \Delta_n >$, the proof is similar to (2) with respect to Fig. 8.4 (g), (h). $\diamondsuit$

The above results tell us that using d_Σ_m or Δ_n as range-domain, there is basically no "continuous image."

A real world concrete example that supports the above results is the RGB-color table. Suppose we have a domain $D = \Sigma_2$ and a range space defined as $D' = \Delta_3$ with length C. See Fig. 8.4 (e). D' has three vertices R, G, and B at the three corners of D. R, G, and B represent "red," "green," and "blue." For any point x' in D', $d(x, R) + d(x, G) + d(x, B) = 2C$. So, if a point $x \in D$ and $f_J(x) = x'$, then we set the color $(r, g, b) = (C - d(x, R), C - d(x, G), C - d(x, B))$ to point x. We know the total color density of each point is the same, $C = r + g + b$. Theorem 8.3.3 tells us that if we wish to color some points in D, it is not always possible to get a "continuous" looking color image using an equal density of colors.

The question that arises is under what condition does this becomes always doable. We know that it is not only "for all $x, y \in J$, $d(x, y) \geq d(f_J(x), f_J(y))$." In practice, one always fixes a color as a base color and adjusts the other two colors to get a continuous looking part of an image.

Definition 8.3.2. *A (point-) connected component of a discrete manifold D is called a discrete sub-manifold S. A sub-manifold is said to be semi-convex if for any two points x, y in S, $d_S(x, y) = d_D(x, y)$.*

The following is not difficult to prove:

Corollary 8.3.2. *If D can normally immerse into D', then any semi-convex sub-manifold of D can also normally immerse into D'.*

Corollary 8.3.3. *A complete graph K_n can normally immerse into any graph, and each graph can normally immerse into K_n.*

There are two computer complexity issues:

(1) Is there any polynomial time algorithm to decide if D is normally immersible into D' ?

(2) More specifically, for a given $< D, D', J, f_J >$, is there any polynomial algorithm that can obtain a gradually varied extension of f_J? We think that the answer is "No."

8.4 Gradually Varied Functionals

In Section 8.2, we introduced the concept of gradually varied mapping. Then we discussed the general consideration of interpolation and extension for gradually varied surfaces in Section 8.3. This section will focus on a special case of the range spaces, Σ_1 or $\{A_1, ..., A_m\}$ (also, it is can be viewed as the integer set integers I). A mapping from D to Σ_1 is called a function. More precisely, let D be a "discretization or net sampling" of a metric space M. $\{A_1, ..., A_m\}$ is a "discretization" of the range of a functional f on M, $f :\rightarrow R$, where R represents the real number. In this case, we want to study, in detail, the gradually varied extension.

After discussion of the theoretical relation between the gradually varied functional and the metric space, we will concentrate mainly on the envelope theorem, uniqueness, and the norm-preserving extension theorem for gradually varied functionals.

Suppose that M is a constructive-compact-metric-space described by Bishop and Bridges [3]. Then there exists a routine which generates a sequence of a finite set $M_1, ..., M_n, ...$, such that M_n is a 2^{-n} net of M. Then M_n is a digital manifold, and the adjacent set of $x \in M$ is

$$\{y|y \in M_n \& w(x, y) < 2^{-n}, \text{ where } w \text{ is the metric of } M \ \}.$$

Let f be a continuous functional on M (f is also a uniform continuous functional [3]). Therefore, for any l, there exists n such that: if $w(x, y) < 2^{-n}$, then $|f(x) - f(y)| < 2^{-l}$.

Now, we construct a functional f_n on M_n as follows. Let $A(s) = s \cdot 2^{-l}$, and set $f_n(x) = A(s)$, where $s = \min\{t|f(x) \leq A(t) \& x \in M_n\}$. Then f_n is a 2^{-n}-uniform approximation of f on M_n.

Lemma 8.4.1. *$f_n : M_n \rightarrow \{A(s)\}$ is gradually varied on M_n.*

Proof: If x and y are adjacent in M_n, then $w(x, y) < 2^{-n}$, and $|f(x)-f(y)| < 2^{-l}$. If $f_n(x) \leq f_n(y)$, then $f(x) \leq f_n(x) \leq f_n(y)$. Thus,

$0 \leq f_n(y) - f_n(x) \leq f_n(y) - f(x) \leq f_n(y) - f(y) + f(y) - f(x) < 2 \cdot 2^{-l}$, so $|f_n(x)-f_n(y)| < 2\cdot 2^{-l}$. Similarly, in case of $f_n(x) > f_n(y)$, we can also obtain $|f_n(x)-f_n(y)| < 2\cdot 2^{-l}$. That is, assume $f_n(x) = A(s)$ and $f_n(y) = A(t)$, then we must have $|t - s| < 2$. Hence, f_n is gradually varied. ♢

(M_n, f_n) is said to be an (n, l)-uniform approximation of (M, f). The process of generating $\{(M(n), f_n), n \in I\}$ is called gradually varied approximating. In calculus, Weierstrass' approximation theorem tells us that for any $\epsilon > 0$ there is a polynomial f_p such that f_p is an ϵ-uniform approximation of f. That is to say, the gradually varied function was reasonably proposed.

In addition to the concept of gradually varied functions (or functionals), n is the function of l. If there is an $m > 0$, such that

$$\lim_{l\to\infty} \frac{n(l)}{l^m} = O(1), \tag{8.12}$$

then m is called the order of the gradually varied function.

For an arbitrary M, f, and $\epsilon > 0$, there is an m so that we can generate an mth-order gradually varied function that is an ϵ-uniform approximation of f.

The gradually varied functional is a special case of the gradually varied surface, but it has some noticeable characteristics. For gradually varied functionals, suppose that p and p' in J satisfy $d(p,p') = d(f(p), f(p'))$. The valuations of each point in the shortest path between p and p' are unique. This path is called non-abundant. In other words, a path is a non-abundant path if and only if it is the shortest path between two points, x and y, and $d(x,y) = d(f(x), f(y))$.

According to Theorem 2.1, we can immediately obtain:

Corollary 8.4.1. *Let J be a non-empty subset of D. If a mapping $f_J : J \to \Sigma_1$ has a gradually varying extension on D, Then the necessary and sufficient condition for the existence of a unique gradually varied extension is that any point a in D is on a non-abundant path whose two ends are in J.*

8.4.1 The Envelope Theorem

The following theorem deals with enveloping, which means that there exists a lower bound and an upper bound in all the gradually varied functions with respect to $< D, \Sigma_1, J, f_J >$.

Theorem 8.4.1. *If J is a non-empty subset of D and a mapping $f_J : J \to \Sigma_1$ has a gradually varying extension on D, then there are two gradually varied functions S_1 and S_2 on D, such that every gradually varied function S is between S_1 and S_2. In other words, for any $x \in D$, $S_1(x) \geq S(x) \geq S_2(x)$.*

Proof: First, for all $x \in J$, let $S_1(x) = S_2(x) = f_J(x)$. For $x \notin J$, get a large enough number t from $\{A_i | i \in Z\}$, let $S_1(x) = A_t$. Test the following inequality for x and each point p in J, i.e.,

$$d(x,p) \geq ? d(S_1(x), S_1(p)),\ \forall p \in J. \tag{8.13}$$

If the above statement is not true, then decrease t by 1, i.e., $t \leftarrow t-1$, and repeat the process of testing. Since gradually varied functions of f_J exist, there must exist a t that satisfies the above equation. In addition, such a valuation satisfies:

$$\exists_{q\in J}((f_J(q) < S_1(x))\ \&\ (d(x,q) = d(S_1(x), S_1(q)))). \tag{8.14}$$

Similarly, get a small enough number s from $\{A_i | i \in Z\}$, let $S_2(x) = A_s$. Test the following inequality for x and each point p in J, i.e.,

$d(x,p) \geq ?d(S_2(x), S_2(p)) \; \forall p \in J.$

If the above statement is not true, then increase s by 1, i.e., $s \leftarrow s+1$, and repeat the process of testing. Since gradually varied functions of f_J exist, there must exist an s that satisfies the above equation. In addition, such a valuation satisfies:

$$\exists_{q \in J}((f_J(q) > S_2(x)) \; \& \; (d(x,q) = d(S_2(x), S_2(q)))). \tag{8.15}$$

For each $x \in D-J$, $S_1(x)$ and $S_2(x)$ represent the upper and lower bound values respectively for any possible gradually varied functions. Thus, if f_D is a gradually varied extension of f_J, then,

$$\forall x \in D\{S_2(x) \leq f_D(x) \leq S_1(x)\}$$

The only thing that remains is to prove that $S_1(x)$ and $S_2(x)$ are gradually varied.

Let x, y be adjacent in D. If $d(S_1(x), S_1(y)) > 1$, we may assume that $S_1(y) < S_1(x)$. According to (4.3) we have a q such that

$$d(y,q) = d(S_1(y), S_1(q)) \text{ and } S_1(q) < S_1(y)$$

but,

$$d(S_1(x), S_1(q)) = d(S_1(x), S_1(y)) + d(S_1(y), S_1(q))$$

Thus,

$$\begin{aligned} d(S_1(x), S_1(q)) &\geq d(S_1(x), S_1(q)) \\ &\geq d(S_1(x), S_1(y)) + d(S_1(y), S_1(q)) \\ &> 1 + d(S_1(y), S_1(q)) \\ &> 1 + d(y,q) \end{aligned}$$

However, we know x and y are adjacent, so

$$d(x,q) \leq d(y,q) + 1.$$

So, a contradiction appears. Thus, $d(S_1(x), S_1(y)) \leq 1$. That is to say, S_1 is gradually varied. In the same way, we can prove that S_1 is gradually varied too. ♢

8.4.2 The Norm-preserving Gradually Varied Extension

In functional analysis, if M is a linear norm(ed) space, then

$$||f|| = \sup_{x \neq 0} \frac{||f(x)||}{||x||} = \sup_{||x||=1} ||f(x)||. \tag{8.16}$$

is called the norm of f.

How to define the norm for a gradually varied function is an interesting question.

Definition 8.4.1. *Let f_D be a gradually varied functional on D. The length of the longest non-abundant path is called the norm of f_D, which is denoted by $||f_D||$.*

$||f_D||$ has geometric meaning. The following is not difficult to prove:

Lemma 8.4.2. *Let D be a discrete space, and f be a gradually varied functional on D. If the path $p_0, ..., p_n$ has the property of $f(p_0) < ... < f(p_n)$, then $p_0, ..., p_n$ is a non-abundant path.*

Theorem 8.4.2. *subset J of D. If for some $n > 0$ we have:*
1) for all p and p' in J, $d(p, p') \geq d(f_J(p), f_J(p'))$;
2) if $d(f_J(p), f_J(p')) = k \cdot n + i$, $0 \leq i < n$ then
a) $d(p, p') \geq k \cdot n + (k - 1)$, $i = 0$, or
b) $d(p, p') \geq k \cdot n + k + i$, $i \neq 0$,
then there exists a gradually varied extension f_D of f_J with $||f_D|| \leq n$. Conversely, 1) and 2) are necessary for a gradually varied extension f_D with $||f_D|| \leq n$.

Proof: First, for all $x \in J$, let $f(x) = f_J(x)$. Select $x \notin J$ and x has an adjacent point $r \in J$. Based on Theorem 8.2.1, f_J must have a gradually varied extension.

(i) Suppose that there is only one choice for $f(x)$ at point x.

(i.a) If $f(x) = f(r)$, then there exists two points $\xi, \eta \in J$ such that $d(x, \xi) = d(f(x), f(\xi))$ and $d(x, \eta) = d(f(x), f(\eta))$ with $f(\xi) < f(x) < f(\eta)$.

For any point $p \in J$, if $d(f(x), f(p)) = kn + i$, this implies

$$d(x, p) < kn + k - 1, \text{ if } i = 0,$$

or

$$d(x, p) < kn + k + i, \text{ if } i \neq 0.$$

Then, we might as well assume $f(p) > f(x)$, and

$$d(x, \xi) = k'n + i', 0 \leq k', 0 \leq i' < n.$$

Thus, if $i = 0$, we have,

$$d(x, \xi) \leq d(p, x) + d(x, \xi) < kn + k - 1 + k'n + i' < (k + k')n + k - 1 + i'.$$

On the other hand, $d(f(p), f(\xi)) \geq d(f(p), f(x)) + d(f(x), f(\xi)) \geq kn + k'n + i' \geq (k + k')n + i'$. Since $p, \xi \in J$, according to the second condition of the theorem, we have $d(p, \xi) \geq (k + k')n + (k + k' - 1) + i'$. It generates a contradiction. So, $d(x, p) > kn + k - 1$ when $i = 0$.

If $i \neq 0$, using the same strategy, we can prove that $d(x, p) < kn+k+i, i \neq 0$.

(i.b) If $f(x) \neq f(r)$, we might as well suppose that $f(r) > f(x)$, then there must be a ξ such that $f(r) > f(x) > f(\xi)$ and $d(f(x), f(\xi)) = d(x, \xi)$

because of the unique choice. We can get the same result as (i.a).

(ii) $f(x)$ has two or more choices for the condition of gradual variation. Because $f(r)$ is fixed, there are at most three choices for $f(x)$. Suppose that two of them are $f(x)$ and $F(x)$. If $d(f(x), F(x)) > 1$, then there is a valid choice in between. So, we can assume that $d(f(x), F(x)) = 1$. Thus, we know $f(r) = f(x)$ or $f(r) = F(x)$. We assume $f(r) = f(x)$.

Assuming that there is $p \in J$, the pair of $(x, f(x))$ and $(p, f(p)$ does not satisfy condition 2) in the theorem. That is, if $d(f(x), f(p)) = k \cdot n + i$, $0 \le i < n$ then

$$\begin{cases} (a)\ d(x,p) < k \cdot n + (k-1) & \text{if } i = 0 \\ (b)\ d(x,p) < k \cdot n + k + i, & \text{if } i \neq 0 \end{cases} \tag{8.17}$$

(In addition, for $p' \in J$, the pair of $(x, F(x))$ and $(p', f(p')$ does not satisfy condition 2) in the theorem. That is, if $d(F(x), f(p')) = k' \cdot n + i'$, $0 \le i' < n$ then

(a) $d(x, p') < k' \cdot n + (k' - 1)$, $i' = 0$, or,
(b) $d(x, p') < k' \cdot n + k' + i'$, $i' \neq 0$.)

Our strategy is to prove that if (8.17) is true, then we can find $F(x)$ that satisfies condition 2) of the theorem.

(ii.a) If $f(p) > f(x)$, raise the assigned value on x by one level, namely $F(x)$. We have $F(x) > f(x)$ and $d(F(x), f(x)) = 1$. For all $\xi \in J$ satisfying $f(\xi) > f(x)$, then $d(F(x), f(\xi)) = d(f(r), f(\xi)) - 1 = kn + i - 1$; $d(x, \xi) \ge d(r, \xi) - 1 \ge kn + k - 1 - 1$ if $i = 0$; $d(x, \xi) \ge d(r, \xi) - 1 \ge kn + k + i - 1$ if $i \neq 0$.

If $i = 0$, then we have

$$d(F(x), f(\xi)) = (k-1)n + (n-1),$$

so

$$d(x, \xi) \ge kn + k - 2 > (k-1)n + (n-1) + (k-1).$$

Thus, $F(x)$ satisfies condition 2) in the Theorem.

If $i \neq 0$, we have

$$d(F(x), f(\xi)) = kn + (i-1),$$

so

$$d(x, \xi) \ge kn + k + (i-1).$$

Whenever $i = 1$ or $i > 1$, $F(x)$ satisfies condition 2) in the Theorem.

For the second part of the theorem, we now consider all $\xi \in J$ satisfying $f(\xi) \le f(x)$. If there is a ξ such that $d(F(x), f(\xi)) = k'n + i'$, then

$$\begin{cases} d(x,\xi) < k'n + k' - 1 & \text{if } i' = 0 \\ d(x,\xi) < k'n + k' + i' & \text{if } i' \neq 0 \end{cases} \tag{8.18}$$

Since $f(\xi) \le f(x) < F(x) < f(p)$,

$$\begin{aligned} d(f(\xi), f(p)) &= d(f(\xi), F(x)) + d(F(x), f(p)) \\ &= d(f(\xi), F(x)) + d(f(x), f(p)) - 1 \\ &= k'n + i' + kn + i - 1 \\ &= (k' + k)n + (i' + i) - 1 \end{aligned} \tag{8.19}$$

We know $d(\xi, p) \le d(p, x) + d(x, \xi)$. Then,
(a) If $i = i' = 0$, according to (8.17) and (8.18), we have

$$d(p, x) < kn + (k - 1).$$

So,

$$d(p, x) \le kn + (k - 1) - 1,$$

$$d(x, \xi) \le k'n + (k' - 1) - 1,$$

and,

$$d(\xi, p) \le d(p, x) + d(x, \xi) \le (k + k' - 1)n + (k + k' - 1) + (n - 3).$$

On the other hand, by (8.19), we have $d(f(\xi), f(p)) = (k'+k-1)n+(n-1)$. Since p, ξ are in J, according to condition 2) in the theorem, we have

$$d(\xi, p) \ge (k' + k - 1)n + (k + k' - 1) + (n - 1)$$

This causes a contradiction.
(b) If $i \ne 0$ and $i' = 0$, we have

$$d(\xi, p) \le (k + k')n + (k + k') + (i - 3).$$

We also have $d(f(\xi), f(p)) = (k' + k)n + (i - 1)$. If $i = 1$, then

$$d(\xi, p) \ge (k' + k)n + (k + k') - 1.$$

If $i > 1$, then

$$d(\xi, p) \ge (k' + k)n + (k + k') + i.$$

So, there is a contradiction.
(c) If $i = 0$ and $i' \ne 0$, we can use the same method in (b) to prove the second part of the theorem.
(d) If $i \ne 0$ and $i' \ne 0$, we have

$$d(\xi, p) \le (k + k')n + (k + k') + (i + i') - 2, \tag{8.20}$$

and

$$d(f(\xi), f(p)) = (k' + k)n + (i + i') - 1.$$

It is obvious that $(i+i')-1>0$, and $(i+i')<2n-1$. Let $(i+i')-1=tn+j$, then $d(f(\xi),f(p))=(k'+k+t)n+j$.

If $j=0$, then $t>0$ and

$$d(\xi,p)\geq(k'+k+t)n+(k+k'+t)-1.$$

However, according to (8.20), we have

$$\begin{aligned}d(\xi,p)&\leq(k+k')n+(k+k')+(i+i')-2\\&\leq(k+k')n+(k+k')+tn-1\\&\leq(k+k'+t)n+(k+k')-1.\end{aligned}$$

This causes a contradiction because $t>0$.

If $j\neq0$,

$$d(\xi,p)\geq(k'+k+t)n+(k+k'+t)+j.$$

But (8.20) tells us,

$d(\xi,p)\leq(k+k')n+(k+k')+tn+j-1\leq(k+k'+t)n+(k+k')+j-1.$ So, there is a contradiction.

To summarize, we have proved that if $(x,f(x))$ and $(p,f(p))$ do not satisfy condition 2) in the theorem, then for $f(p)>f(x)$, we can find $F(x)>f(x)$ and $d(f(x),F(x))=1$ such that $(x,F(x))$ satisfies condition 2). It is not difficult to know that if $f(p)<f(x)$, we just need to decrease one level of $f(x)$ to satisfy condition 2). That is to say, if $f(x)=A_i$, then $F(x)=A_{i-1}$ is the correct choice for the value on x.

We can repeat the above process to assign the value until all points $x\in D-J$ are valued.

We now verify $||f||\leq n$. If $||f||>n$, then there is a non-abundant path with the length of $n+1$. So, there is x,y such that $d(x,y)=d(f(x),f(y))=n+1$. However, according to condition 2), if $d(f(x),f(y))=n+1$, then $d(x,y)\geq n+1+1\geq n+2$. This causes a contradiction. This completes the proof of the first part.

Conversely, if f is gradually varied and $||f||=n$, then we can prove that f satisfies conditions 1) and 2) of Theorem 8.4.2.

Suppose that f is gradually varied and $||f||=n$. Let there exist x,y in D and $d(f(x),f(y))=kn+i$, for $0\leq i<n$. If $d(x,y)\leq kn+k-2$ when $i=0$, then let $x=p_0,p_1,...,p_m=y$, $m<kn+k-1$, be a shortest path between x and y. We might as well assume $f(x)<f(y)$. (If $m=n$, then there is a contradiction.)

Let's examine the first $n+2$ points of the path, $x=p_0,p_1,...,p_{n+1}$–the length of $p_0,p_1,...,p_{n+1}$ is $n+1$. If we have $f(p_0)<f(p_1)<,...,<f(p_{n+1})$, then $||f||>n$. Thus, there must be an i, where $i\leq n$, such that $f(p_i)\geq f(p_{i+1})$. Such a point $f(p_i)$ is called a separating point. The second separating point must appear before p_{2n+1},...., and so forth. That is to say, if we know $d(x,y)\geq kn$, then there must be $k-1$ separating points. So $d(x,y)\geq kn+k-1$. For $i\neq0$, we can prove this statement in the same way. ♢

8.4.3 The discussion of the results of uniqueness, enveloping, and norm-preserving

In practice, the purpose of norm-preserving a gradually varied extension is to simulate the famous Hahn-Banach theorem in functional analysis. Based on Theorem 8.4.2, we can easily obtain:

Theorem 8.4.3. *Let D be a discrete manifold and J be a convex sub-manifold. If the function f_J satisfies the condition of gradual variation, then there is a gradually varied extension of f_J, f_D, such that $||f_D|| = ||f_J||$.*

Even though we do not simulate the Hahn-Banach theorem, Theorems 8.4.2 and 8.4.3 are still meaningful. They state how to get a gradually varied surface without adding the "gradient." The proof of Theorem 8.4.2 also gives an algorithm to construct such a function.

The enveloping theorem, Theorem 4.1, tells us the limitations of the family of gradually varied functions with respect to f_J. We can easily get the following results.

Corollary 8.4.2. *Let S_1 and S_2 be upper and lower envelop surfaces. Then,*

$$S_{m+} = \{< x, S_{m+}(x) > | S_{m+}(x) = A_{\frac{i+j}{2}}, A_i = S_1(x) \& A_j = S_2(x)\},$$

and

$$S_{m-} = \{< x, S_{m-}(x) > | S_{m-}(x) = A_{\frac{i+j}{2}}, A_i = S_1(x) \& A_j = S_2(x)\}.$$

are also gradually varied.

Using the same method, we can get a so-called major family of gradually varied functions based on the $A_{\frac{ai+bj}{a+b}}$ scheme, where a and b are integers.

8.5 Optimal Uniform Approximation and Surface Fitting

The nature of a gradually varied extension is a type of interpolation. When $< J, f_J >$ does not satisfy the condition of gradual variation, we would naturally consider its approximation. In other words, we want to find a gradually varied function f that has the minimum distance to f_J. In numerical analysis, there are two kinds of common approximations: uniform approximation and least squares approximation.

In previous sections, we have described the necessary and sufficient condition of the gradually varied extension. Given a digital manifold D, its a non-empty subset J, and a function $f_J : J \to \{A_i | i \in Z\}$ where $A_{i-1} < A_i$ for all i, we want to find a gradually varied function on D. However, f_J may not satisfy the condition of a gradually varied extension: i.e., for all p and p' in J, $d(p,p') \geq d(f_J(p), f_J(p'))$. In this section, we will design an algorithm for the optimal uniform approximation of f_J.

In the case of a gradually varied approximation, the uniform approximation is to find a gradually varied function f_D such that

$$Max\{d(f_D(p), f_J(p))|p \in J\}$$

is minimum. The least square approximation is to find a gradually varied function f_D such that $\sum_{p\in J}(f_D(p) - f_J(p))^2$ is a minimum.

Obviously, the solutions of the above two problems always exist because of the finiteness of the points in D. The question is how to find an efficient algorithm to obtain the fitting.

This section will give a fast algorithm to solve the uniform approximation problem. For the least squares approximation, we have not solved the problem yet. It may be an NP-hard problem.

This section is only concerned with the uniform approximation. Without loss of generality, we can assume for convenience that $A_s = s$. Also, we denote

(1)$F_i(p) = \{f_J(p) \pm t | t = 0, 1, ..., i\}$,

(2)$F_i(p)/F_i(q) = \{x | x \in F_i(p)$ and there is $y \in F_i(q)$ such that $d(p,q) \geq d(x,y)\}$, and

(3)$F_i^k(p) = \cap_{q\in J} F_i^{k-1}(p)/F_i^{k-1}(q)$, where $F_i^0(p) = F_i(p)$.

Lemma 8.5.1. *(1) For every p and k, $F_i^k(p) \subset F_i^{k-1}(p)$.*
(2) If there exists a k, for all $p \in J$, $F_i^k(p) = F_i^{k+1}(p)$. Thus, for all $N > k$, $F_i^k(p) = F_i^N(p)$.

Proof:

(1) It is obvious.

(2) We might as well assume that $N = k + 2$,

$$\begin{aligned} F_i^{k+2}(p) &= \cap_{q\in J} F_i^{k+1}(p)/F_i^{k+1}(q) \\ &= \cap_{q\in J}\{x | x \in F_i^{k+1}(p) \& \exists y \in F_i^{k+1}(q)(d(p,q) \geq d(x,y)\} \\ &= \cap_{q\in J}\{x | x \in F_i^{k}(p) \& \exists y \in F_i^{k}(q)(d(p,q) \geq d(x,y)\} \\ &= \cap_{q\in J} F_i^k(p)/F_i^k(q) = F_i^{k+1}(p). \ \diamondsuit \end{aligned}$$

Theorem 8.5.1. *If $\exists k \forall p \in J(F_i^{k+1}(p) = F_i^k(p) \neq \emptyset)$, then for every $p \in J$, let $g_J(p) = inf\{F_i^k(p)\}$. Then, g_J satisfies that, for any $p, q \in J$, $d(p,q) \geq d(g_J(p), g_J(q))$ and $\forall_{p\in J}(d(g_J(p), f(p)) \leq i)$. And vice versa.*

Proof: For necessity, let g_J satisfiy the condition of gradual variation and $\max_{x\in J}\{d(g_J(x), f_J(x))\} = i$. Then, for any $p \in J$, we have

$$g_J(p) \in F_i(p)$$

and for any $q \in J$, there is $g_J(q) \in F_i(q)$ with $d(p,q) \geq d(g_J(q), g_J(p))$. So, $g_J(p) \in F_i(p)/F_i(q)$.

Thus, $g_J(p) \in \cap_{q\in J} F_i(p)/F_i(q)$, i.e., $g_J(p) \in F_i^{(1)}(p)$. Assume $g_J(p) \in F_i^{(k-1)}(p)$, then $g_J(p) \in F_i^{(k-1)}(p)/F_i^{(k-1)}(q)$, $\forall q \in J$. Hence, $g_J(p) \in$

$F_i^{(k)}(p)$. Up to now we can conclude that for any $k > 0$, $F_i^k \neq \emptyset$. Because $F_i(p)$ is limited, there must be a k so that $\forall p \in J(F_i^{(k)}(p) = F_i^{(k+1)}(p))$.

For sufficientness, if there is a k, $\forall p \in J(F_i^{(k)}(p) = F_i^{(k+1)}(p) \neq \emptyset)$.

We will use mathematical induction method below. We first want to prove that if $x \in F_i^{(k)}(p) \& y \in F_i^{(k)}(p)$, where $x < y$, and if there is a z, $x \leq z \leq y$, then $z \in F_i^{(k)}(p)$. According to the definition of $F_i(p) = F_i^0(p)$, this is true if $k = 0$, . Assume the statement is valid when $k = t - 1$; we want to prove it is true when $k = t$.

Since $x, y \in F_i^t(p)$, then $x, y \in \cap_{q \in J} F_i^{t-1}(p)/F_i^{t-1}(q)$. So, for any $q \in J$, $x, y \in F_i^{t-1}(p)/F_i^{t-1}(q)$. That is to say, there are $\xi, \eta \in F_i^{t-1}(q)$ such that

$$d(p,q) \geq d(x,\xi) \text{ and } d(p,q) \geq d(y,\eta)$$

If $x \leq z \leq y$, we need to prove that one of the following arguments is true: (1) $d(p,q) \geq d(z,\xi)$, (2) $d(p,q) \geq d(z,\eta)$, or (3) there exists $w \in F_i^{t-1}(q)$ such that $d(z,w) = 0$.

(a) Assume $\eta < \xi$. If $y \geq \xi$, then $d(y,\eta) = d(y,\xi) + d(\eta,\xi)$. So, $d(p,q) \geq d(y,\xi)$ and $d(p,q) \geq d(z,\xi)$. If $y < \xi$, then $x < \xi$. So, $d(x,\xi) \geq d(z,\xi)$. Therefore, $d(p,q) \geq d(z,\xi)$.

(b) Assume $\xi < \eta$. If $\xi \geq z$, then $d(p,q) \geq d(z,\xi)$. Thus, $d(p,q) \geq d(z,\eta)$ if $\eta \leq Z$. Since $F_i^{t-1}(q)$ has all elements between ξ and η according to the induction assumption if $\xi < z < \eta$, there must be an element $w = z$ in $F_i^{t-1}(q)$ such that $d(w,z) = 0$.

Therefore, $z \in F_i^{(t-1)}(p)/F_i^{(t-1)}(q)$. Since q is an arbitrary point in J, we have $z \in F_i^{(t)}(p)$. So, we have proven that the elements of $z \in F_i^{(k)}(p)$ are consecutive.

We know that there always exists a k such that $\forall_{p \in J}(F_i^{(k)}(p) = F_i^{(k+1)}(p))$. For every $F_i^{(k)}(p) \neq \emptyset$, let $m(F_i^{(k)}(p))$ and $M(F_i^{(k)}(p))$ be the minimum and maximum elements, respectively.

If $d(p,q) < d(m(F_i^{(k)}(p)), m(F_i^{(k)}(q))$, we might as well assume $m(F_i^{(k)}(p))$ $< m(F_i^{(k)}(q))$. See Fig. 8.5a. For any $y \in F_i^{(k)}(q)$, we have $d(m(F_i^{(k)}(p)), y) \geq d(m(F_i^{(k)}(p), m(F_i^{(k)}(q)) > d(p,q)$. This is a contradiction because $F_i(k)(p) = F_i(k+1)(p)$. Similarly, we can prove $m(F_i^{(k)}(p)) > m(F_i^{(k)}(q))$. See Fig. 8.5b.

Therefore, for any p, q, we have $d(p,q) \geq d(m(F_i^{(k)}(p)), m(F_i^{(k)}(q))$. So, let $g_J(p) = m(F_i^{(k)}(p))$ for all p, then g_J is the ith-gradually varied approximation. $\diamondsuit$

This theorem can be used to verify the envelope Theorem 8.4.1. According to Theorem 8.5.1, we can first compute g_J, then compute its gradually varied function.

8.6 Fast Algorithms and Gradually Varied Extensions

We will use a computer to generate fast algorithms for gradually varied extensions. An algorithm for the uniform approximation is shown below:

Step 1 Initially, let $i = 1$, and compute $F_i(p)$, $F_i(p)/F_i(q)$, and $F_i^k(p)$.

Step 2 Check if the condition of Theorem 3.1 is satisfied. If not, $i \leftarrow i+1$ and go to step 1.

Step 3 Let $g_J(p) = inf\{F_i^k(p)\}$ for each $p \in J$.

Step 4 Find a gradually varied extension f_D of g_J.

A procedure for generating a gradually varied extension is given below:

Procedure A: Given a digital manifold D and its subset J, and given $f_J : J \rightarrow \{A_1, ..., A_m\}$, where $A_1 < ... < A_m$, this procedure is to obtain a gradually varied function $f : D \rightarrow \{A_1, ..., A_m\}$ or state that there is not any gradually varied function f for f_J.

```
BEGIN
      FOR (every pair p, p' in D) DO
         compute d(p, p') by Floyd's algorithm [6];
      FOR (every pair p, p' in J) DO
         IF (d(p, p') < d(f_J(p), f_J(p'))) THEN {there is nof and halt;}
      D_0 := J;
      FOR (every r ∈ D_0) DO { f(r) := f_J(r); }
  5   IF (D_0 = D) THEN
         output f and halt;
      ELSE
         BEGIN
            choose x in D − D_0 such that x has an adjacent point
            r ∈ D_0. (without loss of generality assume f(r) = A_i);
         END
      f(x) := A_i;
      FOR (every p ∈ D_0) DO
         BEGIN
            IF( d(x, p) < d(f(x), f(p)) ) THEN
               IF ( f(x) > f(p) ) THEN ( f(x) := A_{i−1} ) ELSE (
f(x) := A_{i+1} );
         END
      D_0 := D_0 ∪ {x};
      GOTO 5;
END
```

In application, we use the algorithm to fit surfaces in seismic data processing. In [24], we show the results of using the algorithm to garner a series of interfaces in a stratum in an area of the southwest of China. We also developed a C++ program to fit gradually varied surfaces based on above algorithm. See Fig. 8.6.

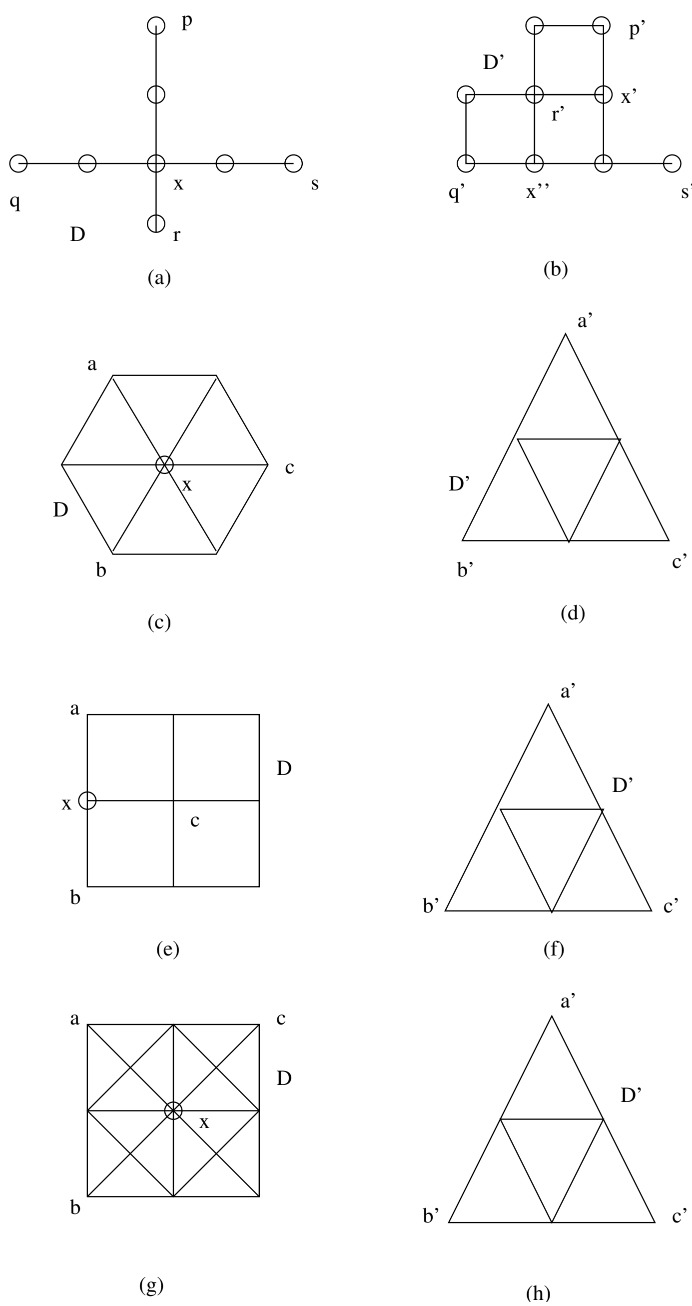

Fig. 8.4. The example that shows no gradually varied extension.

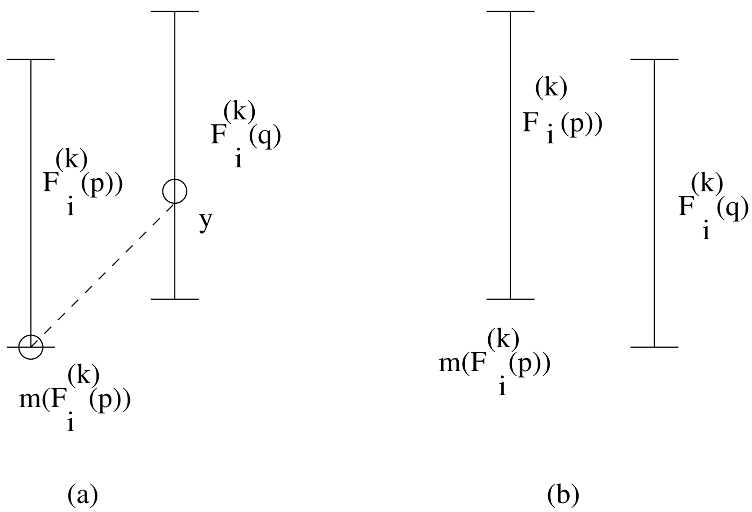

Fig. 8.5. Two cases of $F_i^{(k)}(p))$ and $(F_i^{(k)}(q)$.

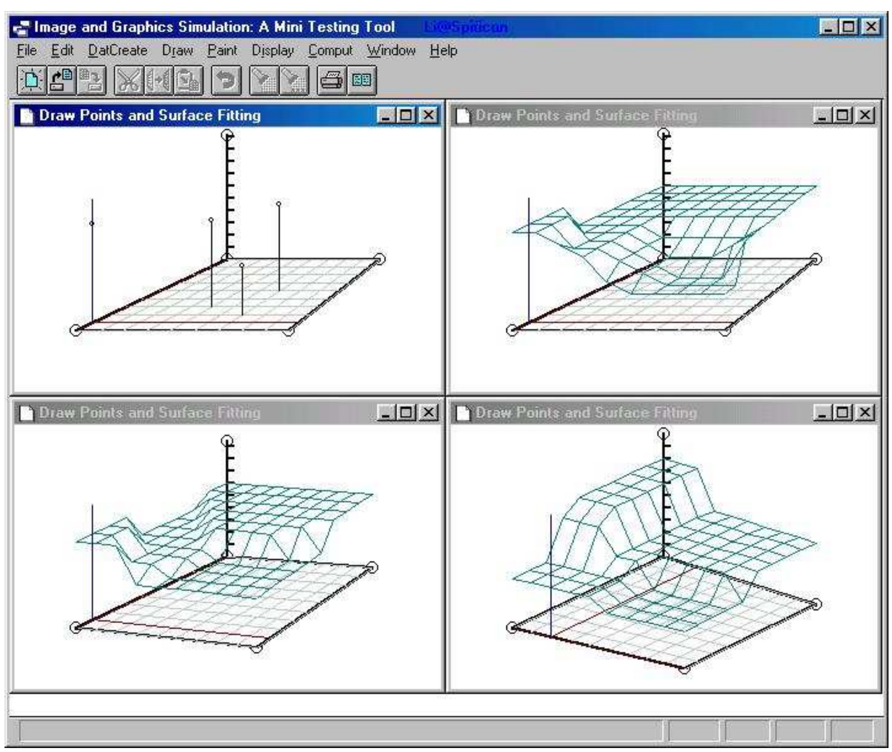

Fig. 8.6. Gradually varied surface fitting