5. Digital Surfaces: Advanced Topics in 3D

In this chapter, we will first present the classification theorem for digital surfaces in direct adjacency in three-dimensional space that was given by Chen and Zhang in 1993 [29]. The classification theorem deals with the categorization of simple surface points. It states that there are exactly six different types of simple surface points [29]. Francon (1995) remarked on the same six surface points in [33].

Second, we will present a mathematical proof for the equivalence between the two definitions of digital surfaces given by Morgenthaler and Rosenfeld [57] and Chen and Zhang [28] in direct adjacency.

Third, on the basis of the classification theorem and Euler formula on the planar graph, we will show the corner point theorem: Any simple closed surface has at least eight corner points, where a corner point of a closed surface is a point in the surface which has exactly three adjacent points in the closed surface. Another result reported in this book is that any simple closed surface has at least fourteen points.

Finally, we shall discuss why Morgenthaler and Rosenfeld's surfaces are not complete as well as other issues on general digital surfaces.

5.1 The Classification of Digital Surface Points

This section presents some detailed results on simple closed surface defined by Morgenthaler and Rosenfeld [57]. Chen, Cooley, and Zhang proved a theorem for classification of simple surface points in direct adjacency [29][23]. This theorem states that there are exactly six types of simple surface points in simple closed surfaces. We called this theorem the classification theorem. In this section, we will prove that there are exactly six classes of so-called regular inner surface points (in terms of parallel-move based definition of digital surfaces). In the next section, we will prove the equivalence between this definition and the definition of the digital surfaces given by Morgenthaler and Rosenfeld [57].

5.1.1 Simple Surface Points and Regular Inner Surface Points

The definition of parallel-move based surfaces is simple and intuitive. The question is "what is the relationship between such a digital surface and a Morgenthaler-Rosenfeld surfaces?" This section will show that a closed regular surface is precisely a Morgenthaler-Rosenfeld simple surface.

In order to establish the relationship between Morgenthaler-Rosenfeld's simple surfaces and our parallel-move based surfaces, we need to introduce the concept of regular surface points.

If p is a point of a parallel-move based surface S, p is regular if all of S's surface-cells including p are line-connected in S. If p is both inner and regular, then p is called a regular inner surface point.

To deal with general cases, we can expand the meaning of a regular surface point to any (point-)connected set as follows.

Definition 5.1.1. *Let S be a connected subset of Σ_3. Assume $p \in S$ and $S(p) = S \cap (N_{27}(p) \cup \{p\})$. p is called a regular surface point of S if:*

(1) Each line-cell in S containing p has at least 1 and at most 2 parallel-moves in $S(p)$.

(2) Any two surface-cells containing p in S are line-connected in $S(p)$.

(3) $S(p)$ does not contain any $3D$-cell.

We say p is a regular inner surface point if p is a regular surface point and each line-cell containing p has exactly two parallel-moves in $S(p)$.

Theorem 5.1.1. *A Morgenthaler-Rosenfeld simple surface point is a regular inner surface point. That is, a regular closed surface is a Morgenthaler-Rosenfeld surface.*

Theorem 5.1.1. shall be used to prove the classification theorem, but the proof of its own will be given in Section 5.2.

5.1.2 Geometric Equivalence in 3D

In order to explore the structure of simple closed surfaces, we should know the number of different types of simple surface points. Classification theorem presented in this section states that there are exactly six different types of simple surface points.

Definition 5.1.2. *Let S and R be two subsets in $N(27,p) = N_{27} \cup \{p\}$ and p be a point in both S and R. S and R are geometric equivalent if and only if there is a one-to-one mapping $f : S \to R$ which satisfies:*

(1) $f(p) = p$,

(2) $d(x,y) = d(f(x), f(y))$, d is the distance for 6-connectedness.

(3) $D(x,y) = D(f(x), f(y))$, D is the distance for 26-connectedness.

Where x and y in S and the distance means the length of the shortest path.

It is easy to see that the conditions (1) and (2) are necessary in Definition 5.1.2 for this equivalence. It is not obvious that the condition (3) is necessary. However, without condition (3), S and R in Fig. 5.1 are equivalent.

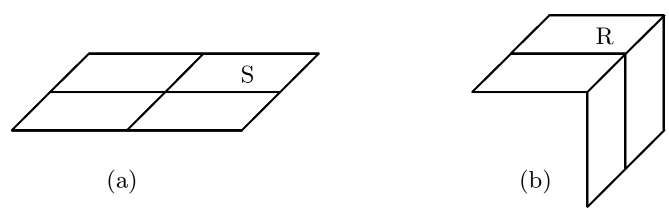


Fig. 5.1. Simple surface points each of which has four adjacent points.

Lemma 5.1.1. *The geometric equivalence relation is a mathematical equivalence relation.*

5.1.3 The Theorem of Classification of Simple Surface Points

A simple surface point p concerns with the point p and its surrounding points. The geometric equivalence relation described in Definition 5.1.2. can classify all $N(27, p)$'s subsets with point p into a number of geometric equivalence classes. Among these classes, only a few of them make p as a simple surface point. Only 6-adjacent simple surface points are considered in this Chapter. Based on the geometric equivalence, all simple surface points will be classified.

Lemma 5.1.2. *If p is a simple surface point, then each line-cell containing p in $N(27, p)$ has exactly two parallel-moves; any two surface-cells are line-connected in $N(27, p)$, and there is no 3D-cell in $N(27, p)$.*

Therefore, p is a regular inner surface point when p is a simple surface point. According to the two lemmas mentioned above, we can obtain:

Theorem 5.1.2. *There are only 6 types of simple surface points that are not geometrically equivalent to each other (see Figure 5.2).*

Proof We can start with a point p and one of its directly adjacent points p' because an isolated point can not be a simple surface point. According to Lemma 5.1.2 and geometric equivalence, the line-cell (p, p') must have two parallel-moves. So, they derive two cases in Fig. 5.3, and nothing else.

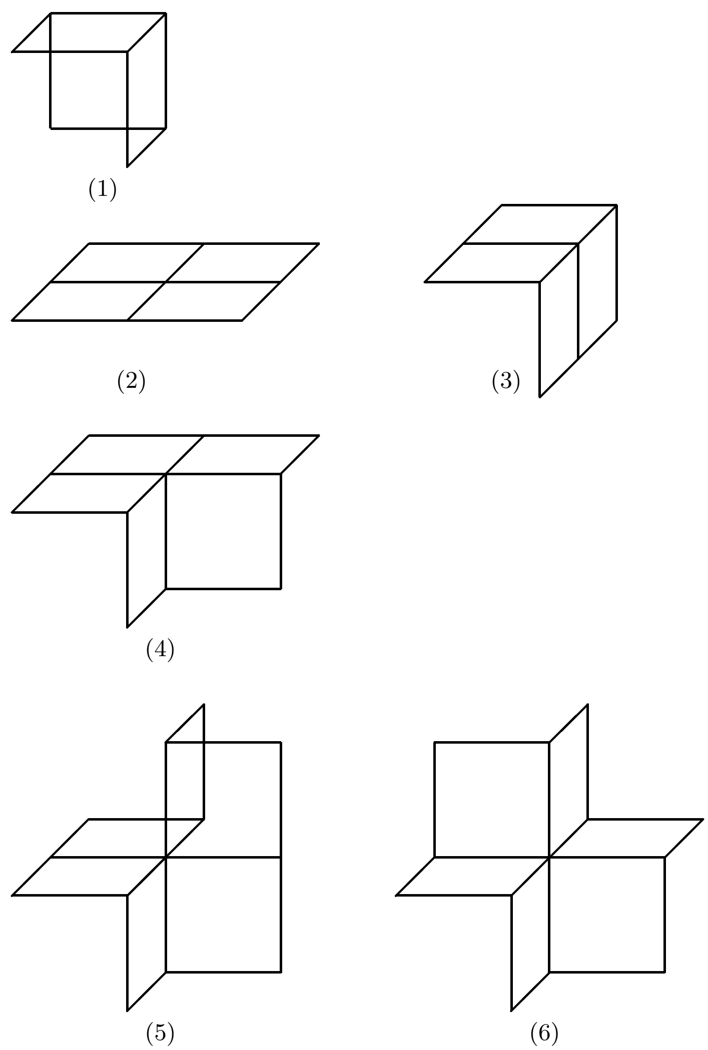


Fig. 5.2. All types of simple surface points.

Thus, a simple surface point with three surface-cells in $N(27, p)$ can be derived from the case (b) shown in Fig. 5.3. It is the case (1) in Fig. 5.2. Because each line-cell must have exactly two parallel-moves, from Fig. 5.3, we can develop the three cases with 3 surface-cells in Fig. 5.4. These are the only possibilities to be simple surface points without duplication under the geometric equivalence relation.

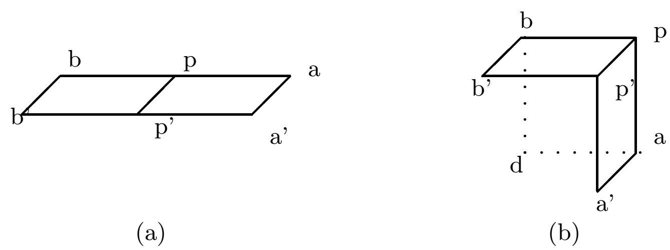


Fig. 5.3. Two cases derived from line-cell (p, p') .

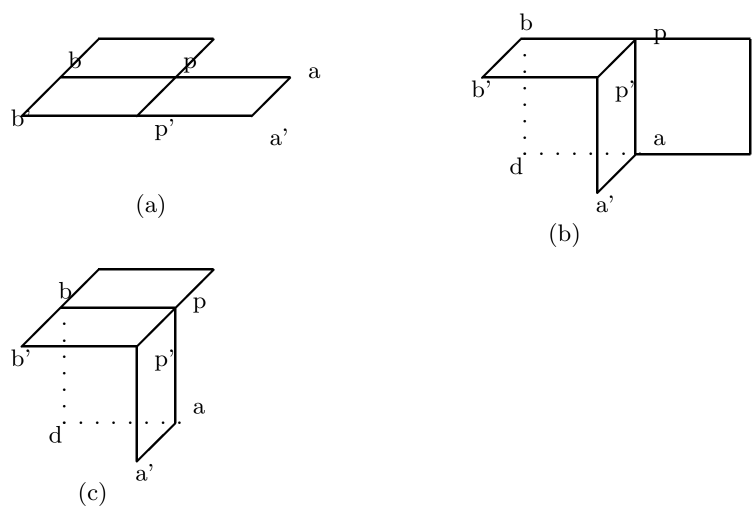


Fig. 5.4. Three cases derived from Fig. 5.3.

Continue the derivation, we can develop the 6 cases with 4 surface-cells from Fig. 5.4, see Fig. 5.5.

Therefore, we get case (2) and case (3) in Fig. 5.2 from case (a) and case (c) in Fig. 5.5. Again, we can develop the 6 cases with 5 surface-cells in Fig. 5.6 from Fig. 5.5.

Next, we get case (4) in Fig. 5.2 from case (e) in Fig. 5.6. We can see that point p in case (d) cannot generate a simple surface point. Cases (a),(b) (c) and (f) in Fig. 5.6 have only one choice to be a simple surface point. When we add a surface-cell to cases (a), (c) or (f), we get the simple surface point that is the same as case (5) in Fig. 5.2. When we add a surface-cell to case

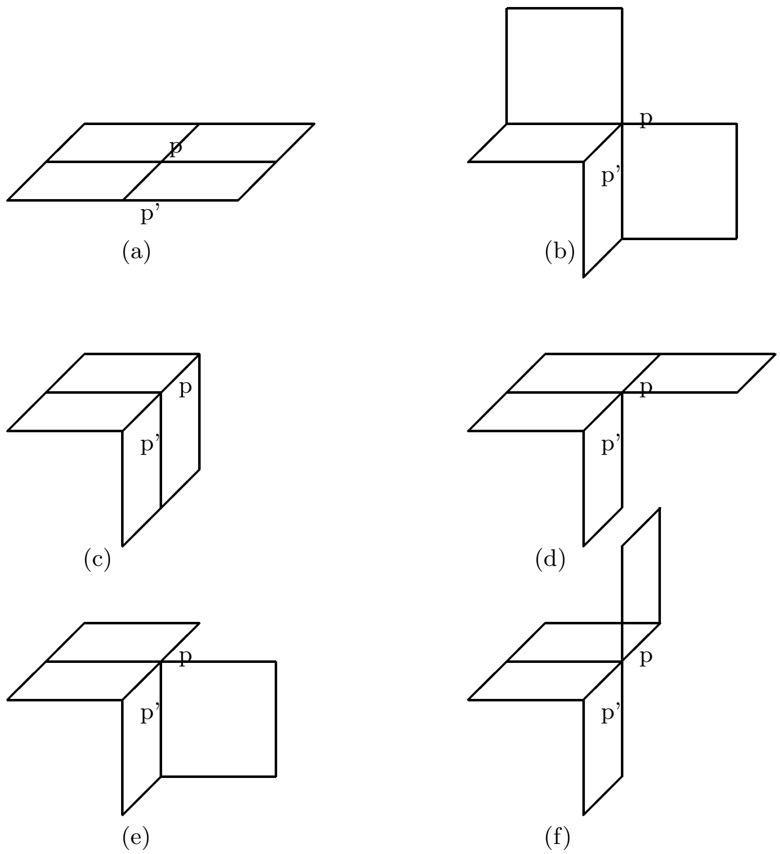


Fig. 5.5. Six cases derived from Fig. 5.4.

(b) in Fig. 5.6, it becomes case (6) in Fig. 5.2. ⋄

5.2 The Equivalence between Two Definitions of Digital Surfaces

This section presents a proof of the equivalence of two digital surface definitions. One of the definitions was developed based on the simple surface points given by Morgenthaler and Rosenfeld, and the other was built on the parallel-move concept provided by Chen and Zhang in direct adjacency.

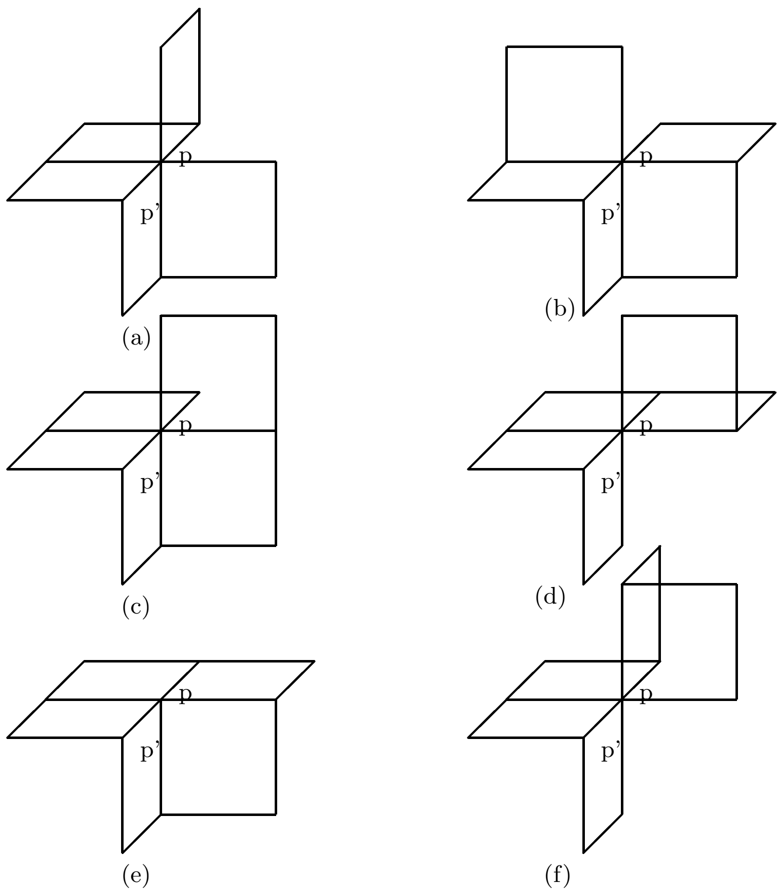


Fig. 5.6. Six cases derived from Fig. 5.5 (b), (d), (e), and (f).

5.2.1 The Theorem of Equivalence

Although most surface points are regular, we show two examples for non-regular surface points in Fig. 5.7.

Theorem 5.2.1. *A simple surface point of a set $S \subset \Sigma_3$ is a regular inner surface point, and vice versa.*

Proof: For clarification, we separate the proof into two parts. In the first part we prove that if p is a simple surface point, then p is also a regular inner surface point. In the second part we prove that if p is a regular inner surface point, then p is a simple surface point.

Proof of part 1: Suppose that p is a simple surface point as reviewed in Section 4.1.1. We want to show the following:

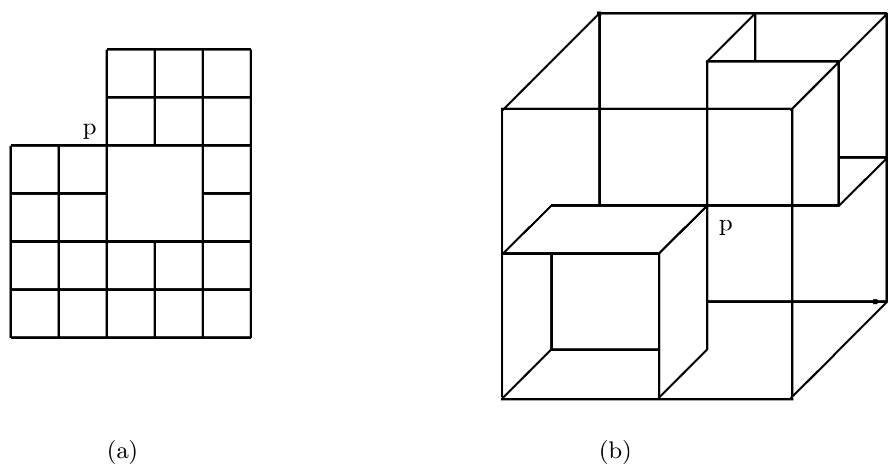


Fig. 5.7. Examples for non-regular points: (a) Non-regular point p; (b) Inner but non-regular surface point p

1) $S(p) = S \cap (N_{27}(p) \cup \{p\})$ does not have any 3D-cell,

2) Each line-cell containing p in S has exactly two parallel moves in $S(p)$, i.e., p is inner, and

3) Any two surface-cells containing p in S are line-connected in $S(p)$.

First, suppose p is a simple surface point in S. Obviously, p cannot be a corner point of any 3D-cell in S, hence we establish statement 1).

For 2), to begin with, p has three or more (directly) adjacent points in $S \cap N_{27}(p)$. Otherwise, $\bar{S} \cap N_{27}(p)$ is connected, so p is not a simple surface point in accordance with condition (2) of its definition in Section 4.1.1.

We define here a grid plane is a set of all points with a fixed z $P_z = \{(x, y, z)|(x, y, z) \in \Sigma_3\}$, all points with a fixed y $P_y = \{(x, y, z)|(x, y, z) \in \Sigma_3\}$, or all points with a fixed x $P_x = \{(x, y, z)|(x, y, z) \in \Sigma_3\}$.

Next, let $p' \in S$ be an arbitrary adjacent point of p. therefore, there is a grid plane (such as $plane1$, $plane2$ or $plane3$ in Fig. 5.8 which contains p in $N_{27}(p)$ and does not contain p'. Thus, all directly and indirectly adjacent points of p' are in one side of the grid plane (including the plane). Dependent upon the third condition of the definition of simple surface points, each $c_1(p)$ and $c_2(p)$, which are defined in Section 4.1.1, have one point in the side of the plane because they must be indirectly adjacent to p'. Also, all of the parallel-moves of line-cell $\{p, p'\}$ are in the side of the plane.

If $\{p, p'\}$ has no two parallel-moves in S, then all of the points which are in $\bar{S} \cap N_{27}(p)$ and are in the side of the plane are connected; thus, $c_1(p)$ and $c_2(p)$ are connected. So, $\{p, p'\}$ has two or more parallel-moves.

Now, we prove line-cell $\{p, p'\}$ has no more than two parallel-moves. In contrast, suppose $\{p, p'\}$ has three parallel-moves; without loss of generality, we let the three parallel-moves $\{q, q'\}$, $\{r, r'\}$, and $\{s, s'\}$ of $\{p, p'\}$ be described in Fig. 5.8:

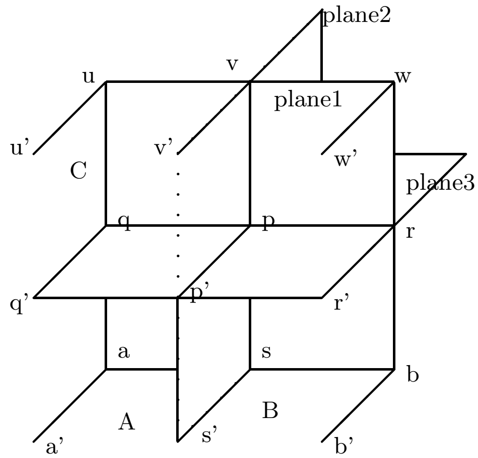


Fig. 5.8. S in $N_{27}(p)$.

Suppose we are sure that $\{p, p', q, q', r, r', s, s'\}$ are in S. We let A be the set whose elements are in $\{a, a'\}$ but not in S, and B be the set whose elements are in $\{b, b'\}$ but not in S. Because $S \cap N_{27}(p)$ does not have any 3D-cells, A and B are not empty.

Since p is a simple surface point; A and B, A and C, or B and C must be indirectly connected according to the second condition of the definition of simple surface points in Section 4.1.1. Using the same reasoning for A and C, and B and C, we only need to prove: p is no longer a simple surface point when A and B, or A and C are indirectly connected.

(i) If A and B are indirectly connected, then $a \in A$ and $b \in B$; otherwise, A does not connect with B in $N_{27}(p) \cup p$. We know that a and b are indirectly connected. Meanwhile, every s's indirectly adjacent point in $N_{27}(p) \cup p$ is below plane3, and two of them are contained in $c_1(p)$ and $c_2(p)$ respectively. Thus, all of s's indirectly adjacent points must be indirectly adjacent to a or b. Then $c_1(p)$ and $c_2(p)$ defined in Section 4.1.1 are indirectly connected, so p is not a simple surface point.

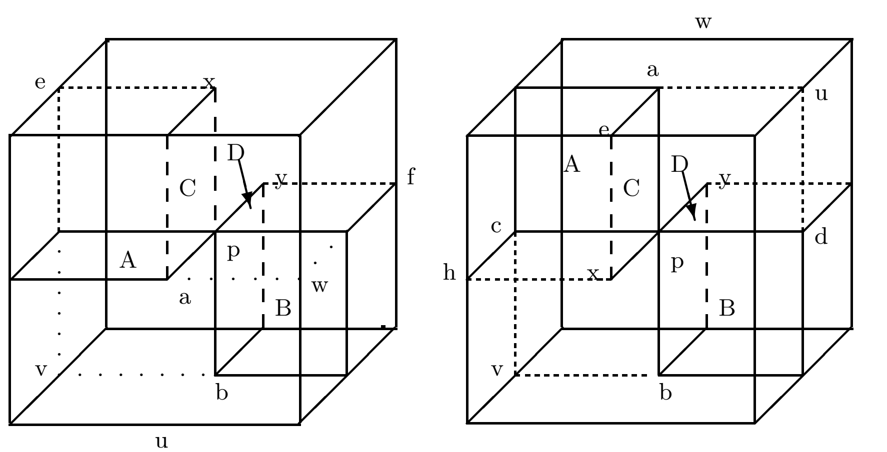
(ii) If A and C are indirectly connected, a must belong to A. We may suppose that $A \cup C \subset c_1(p)$ and $B \subset c_2(p)$. We now must discuss the following two cases.

(ii.a) If $b \in S$, then $c_2(p) = b'$. Thus, q cannot be indirectly adjacent to b, i.e., $c_2(p)$. According to the third condition of the simple surface points' definition, p is not a simple surface point.

(ii.b) If b is not in S, i.e., $b \in c_2(p)$; then we can see if q connects to b and r connects to a, they must pass the plane 2. On the other hand, if $\{u, v, w\} \subset S$, then there is a point of $c_1(p)$ in $\{u', v', w'\}$ based on no 3D-cell in S. However, the point and A cannot be indirectly connected in $N_{27}(p)$, that is, A and C are not indirectly connected. Thus, there must be a point δ in u, v, and w such that it is in $c_1(p) \cup c_2(p)$. If δ is in $c_1(p)$ then each point in plane 2 indirectly connects with a or δ. q cannot indirectly connect with the point b. If δ is in $c_2(p)$ then r cannot indirectly connects with the point a. According to the third condition of the simple surface point definition, p is not a simple surface point.

Therefore, we have proven statement 2. We now prove statement 3 to complete part one of the proof. Statement 3 says that any two surface-cells of $S(p)$ are line-connected if p is a simple surface point.

Actually, there are only two cases for two surface-cells A and B including p in $N_{27}(p) \cup p$ which are not line-adjacent (See Fig. 5.9). In the following we show that these two surface-cells A and B are line-connected in $N_{27}(p) \cup p$ when p is a simple surface point and both A and B are in S.



Case (a) Case(b)

Fig. 5.9. Two cases in which surface-cells are not line-adjacent.

For case (a), if one of u, v, or w is in S, then surface-cells A and B are line-connected. Otherwise, because each (p, a) and (p, b) have two parallel-moves, C and D must be in S. By the same reasoning, points e and f are in S, so we can see that the p is not a simple surface point because $\bar{S} \cap N_{27}(p)$ has three indirectly connected parts.

For case (b), if u and v are in S, then A and B are line-connected. Otherwise, C and D ,or $\{a, e, x, p\}$ and $\{a, p, y, w\}$ must be in S; so we have a case like case (a). Thus, p is not a simple surface point if A and B are not line-connected.

We have thus proven statement 3. To summarize, p is a regular inner surface point if p is a simple surface point.

Proof of part 2: Suppose p is a regular inner surface point. We want to show that p is a simple surface point, i.e., the following three statements are true:

1) $S \cap N_{27}(p)$ has exactly one component adjacent to p, denote this component A_p.

2) $\bar{S} \cap N_{27}(p)$ has exactly two 26-connected components, c_1 and c_2, 26-adjacent to p.

3) If $q \in S$ and q is adjacent to p, then q is 26-adjacent to both c_1 and c_2.

Our strategy for proving part two is different from part one's proof. We just enumerate all possibilities in which a regular inner surface point p can appear, and we then prove that all possible regular inner surface points are simple surface points.

We know if p is a regular surface point of S, then p is not a corner point of any 3D-cell which is in S. Because of the (point-) connectivity of S, p has an adjacent point denoted by p'. Also, $\{p, p'\}$ has two parallel-moves, denoted by $\{a, a'\}$ and $\{b, b'\}$; therefore, both a and b are also adjacent to p.

Suppose the surface-cell that is formed by $\{p, a\}$ and $\{p, b\}$ is in S, then there are three surface-cells that are line-connected to each other in $S(p)$. Therefore, they can be illustrated as shown in Fig 5.10.

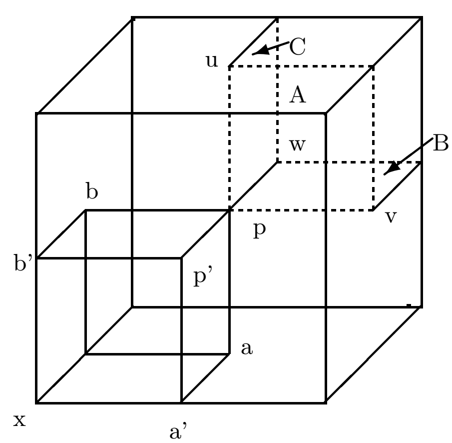


Fig. 5.10. Not a regular surface point.

Here x must not be in S as p is not a corner of any 3D-cell. If there is a point u, v, or w that is adjacent to p; then A, B or C must be in S because any line-cell has exactly two parallel-moves. In this case, there exist two surface-cells which are not connected in $N_{27}(p) \cup \{p\}$, so p is not a regular surface point. Thus, there is no such point u, v, or w which is adjacent to p, so we can see p satisfies the definition of a simple surface point.

On the other hand, if the surface-cell formed by $\{p, a\}$ and $\{p, b\}$ is not in S. We can derive only two different cases as shown in Fig 5.11.

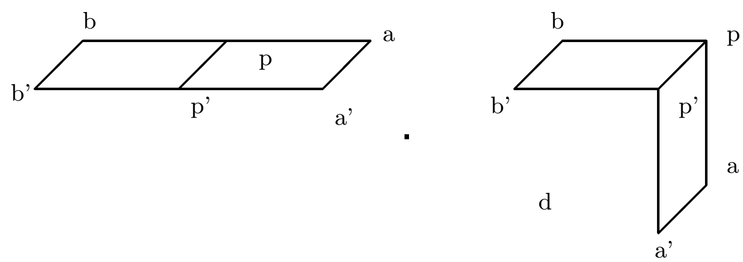


Fig. 5.11. Not a regular surface point.

We now illustrate all possible developments based on the above cases, meaning that p is kept as a regular inner surface point. We only consider the points that are adjacent to p and the surface-cells that contain p. It is straightforward to see that there is only one possible case for a regular inner surface point with exactly three surface-cells.

(i) If there are exactly 4 surface-cells in $S(p)$, we have only the following two possible cases keeping p as a regular inner surface point (Fig. 5.12). We can see that p in either (a) or (b) is a simple surface point.

(ii) If there are 5 or more surface-cells in $S(p) = S \cap (N_{27}(p) \cup p)$, then only two cases which contain 3 surface-cells can be developed to generate different results. The two cases are given in Fig. 5.13.

Each (a) and (b) in Fig. 5.13 has 3 possible ways for adding one more surface-cell. Some of which are overlap. We can reduce such cases to 4 distinct cases as shown in Fig. 5.14.

Next, we continuously add new surface-cells to (a), (b), (c), and (d) of Fig. 5.14. and maintain p as a regular inner surface point. We arrive at the following seven cases shown in Fig. 5.15. We can see that (e) of Fig. 5.15. already arrives at the final state, where it has only five surface-cells in $S(p)$. On the other hand, (c) also arrives at the final state because it cannot be a regular inner surface point.

We also can see that there is only one possible choice to add a surface-cell onto (a), (b), (d), (f), and (g) in Fig. 5.15. After adding a surface-cell,

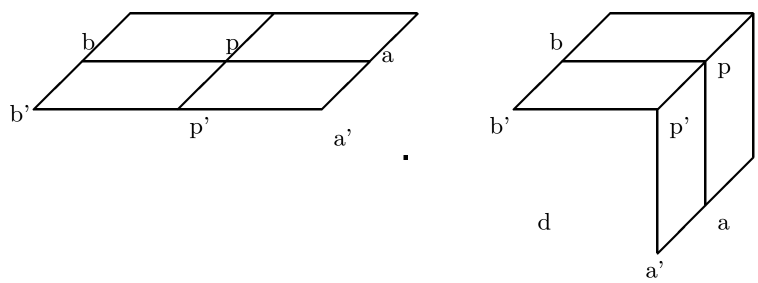


Fig. 5.12. $S(p)$ has exactly four surface-cells.

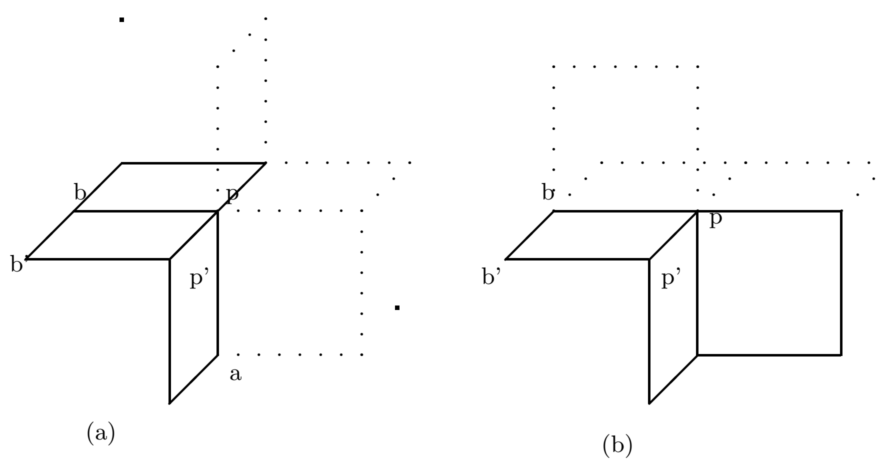


Fig. 5.13. Two cases that can be developed.

(a), (b), (d), (f), and (g) are deduced into 2 cases shown in Fig. 5.16. Each of which have 6 surface-cells including p, where p is a regular inner surface point.

Finally, we shall explain , when p is a regular surface point in S, why there are no 7 or more surface-cells including p in $S \cap (N_{27}(p) \cup \{p\})$ that can make a simple surface point. We know p has 6 adjacent points in $N_{27}(p) \cup \{p\}$; in other words, there are 6 line-cells including p. If a surface-cell A including p is in S, then A contains 2 of the 6 line-cells. When S has 7 surface-cells, there must exist a line-cell which is included by 3 surface-cells. Therefore, S is not a surface.

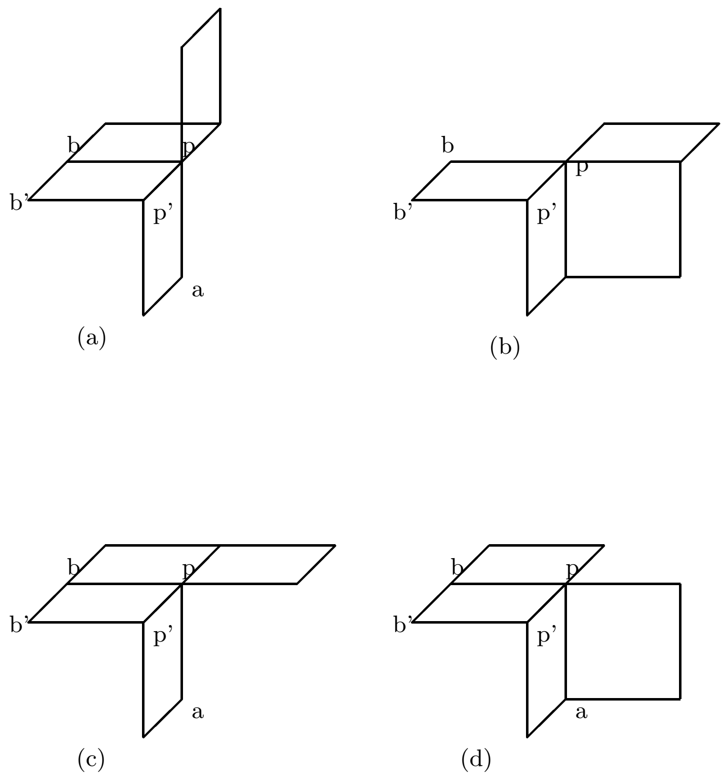


Fig. 5.14. Four cases derived from Fig. 5.13.

From the preceding 5-step process, we obtain three types of regular inner surface points contained by 5 or 6 surface-cells. Considering Fig. 5.11. and Fig. 5.12., we have one regular inner surface point with 3 surface-cells and two regular inner surface points with exactly 4 surface-cells. There are only 6 possibilities for p to be a regular inner surface point. We can see that all of the three kinds of the regular inner surface points satisfy the definition of the simple surface points. We now have completed the proof of part 2. Therefore, every regular inner surface point is a simple surface point. $\diamond$

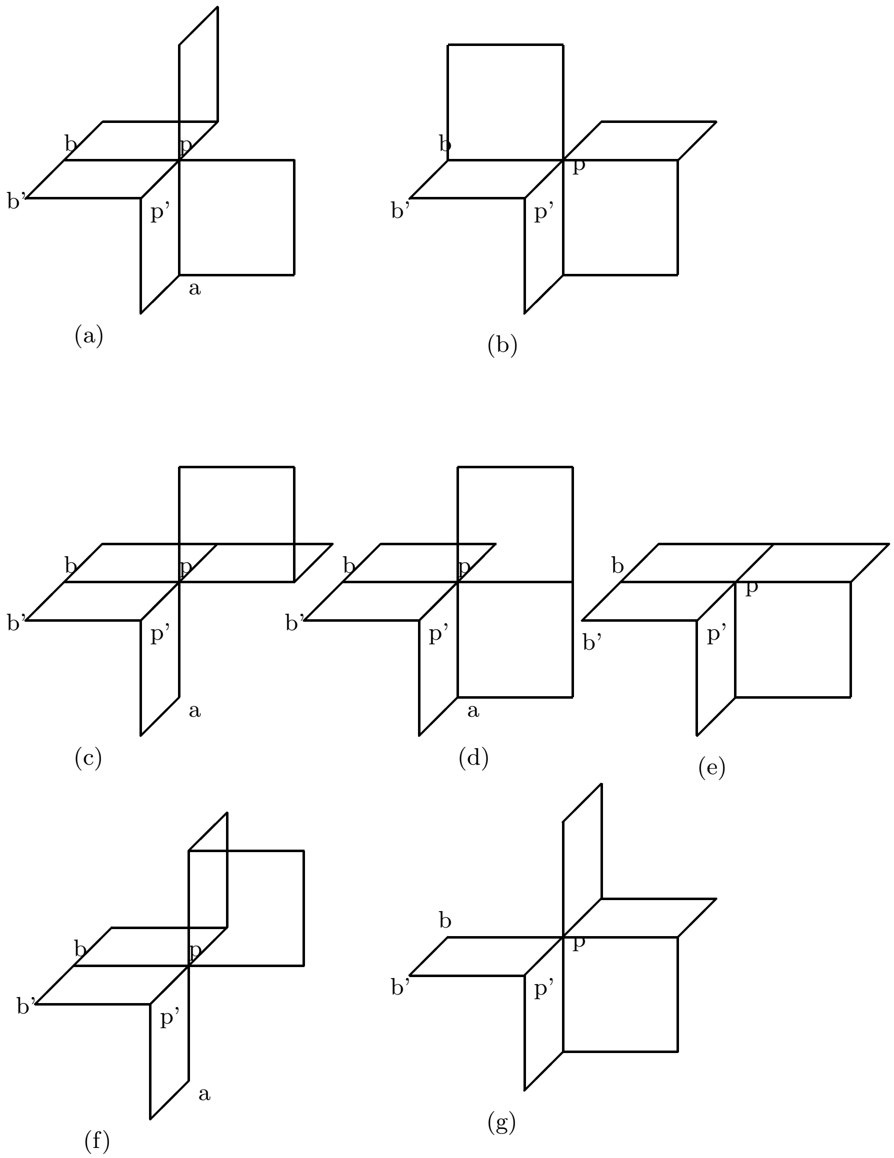


Fig. 5.15. Seven cases derived from Fig. 5.14.

Since none of the discussions used the first condition of Definition 4.1.2,

Corollary 5.2.1. *The second and third conditions of the definition of Morgenthaler and Rosenfeld's surface imply the first condition.*

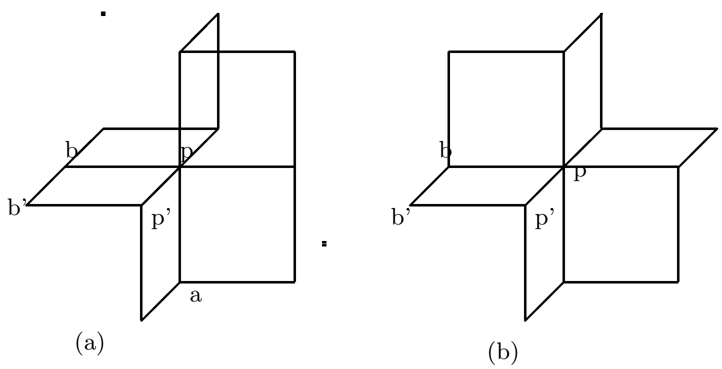


Fig. 5.16. Two cases deduced from Fig. 5.15.

5.2.2 More About Regular Surface Points

In the above section, we defined a regular surface point based on parallel-move based surfaces. We have obtained exact six types of regular inner surface points. In practice, it may be useful to extract all simple surface points in a set S by matching the six types of simple surface points. Theoretically, to decide if S is a simple surface need not to test all three conditions of Definition 4.1.2 or to match six types of regular inner surface points.

A Morgenthaler-Rosenfeld's simple surface S is closed, and each point is a simple surface point. A fast algorithm can be developed by testing if S is a parallel-move based closed surface first, and then testing if S contains the case of Fig 5.7. (b).

In addition, the definition of simple surface points cannot be used to deal with boundary points. However, the disadvantage of the definition of regular surface points is how to expand it to deal with indirect adjacency. This issue was discussed in [13]. Comparing with Kong and Roscoe's results [48], we found that there are some similarities between Definition 5.1.4 and Proposition 6 in [48]. (See Section 5.4.) However, Definition 4.1.4 is still much general, simpler and intuitive.

5.3 Digital Surfaces and Euler's Theorem

Based on classification theorem in Section 5.1 and Euler's Formula on planar graph [39], in this section, we prove a global theorem: Any simple closed surface at least has eight corner simple surface points.

Euler's formula of the planar graph states: If $G = (V, E)$ is a planar graph and F is the set of faces in G, then $|F| + |V| = |E| + 2$. Let S be a simple closed surface. Because there is no complete bipartite graph $K_{3,3}$ and complete graph K_5 in S, so S is a planar graph. Thus,

Lemma 5.3.1. *S is a planar graph if we view S as a graph $G = (V, E)$, where $V = S$ and $(p, p') \in E$ iff p and p' are adjacent in S.*

Let p be a point of S. p is called i-point if p is 6-adjacent to i points in $S \cap N(27, p)$. Let $M_3(S)$, $M_4(S)$, $M_5(S)$, and $M_6(S)$ be the sets of the 3-point (also called corner point), 4-point, 5-point, and 6-point, respectively. There is no 7-point in a simple surface S [23].

Lemma 5.3.2. *(1)* $|V| = |M_3(S)| + |M_4(S)| + |M_5(S)| + |M_6(S)|$;
(2) $|E| = (3|M_3(S)| + 4|M_4(S)| + 5|M_5(S)| + 6|M_6(S)|)/2$;
(3) $|F| = (3|M_3(S)| + 4|M_4(S)| + 5|M_5(S)| + 6|M_6(S)|)/4$.

Proof: For (1), according to Theorem 5.1.2, $M_0(S)$, $M_1(S)$, $M_2(S)$ and $M_7(S)$ etc. can not be the simple surface points. Therefore (1) is true.

For (2), we just consider each vertex's contribution to the edges E in planar graph G. In $G = (V, E)$, every point in $M_3(S)$ is connected with 3 edges, every point in $M_4(S)$ is connected with 4 edges, every point in $M_5(S)$ is connected with 5 edges, and every point in $M_6(S)$ is connected with 6 edges. On the other hand, every edge has two end points. So,

$$|E| = (3|M_3(S)| + 4|M_4(S)| + 5|M_5(S)| + 6|M_6(S)|)/2.$$

For (3), we also consider each vertex's contribution to the faces F in G. In $G = (V, E)$, every point in $M_3(S)$ is connected with 3 faces, every point in $M_4(S)$ is connected with 4 faces, every point in $M_5(S)$ is connected with 5 faces, and every point in $M_6(S)$ is connected with 6 faces. On the other hand, every face has four end points. So,

$$|F| = (3|M_3(S)| + 4|M_4(S)| + 5|M_5(S)| + 6|M_6(S)|)/4.$$

We complete the proof. $\diamond$

Lemma 5.3.3. $|M_3(S)| = 8 + |M_5(S)| + 2|M_6(S)|$.

Proof: According to the Euler's formula, we have:

$$|M_3(S)| + |M_4(S)| + |M_5(S)| + |M_6(S)| +$$

$$(3|M_3(S)| + 4|M_4(S)| + 5|M_5(S)| + 6|M_6(S)|)/4 =$$

$$(3|M_3(S)| + 4|M_4(S)| + 5|M_5(S)| + 6|M_6(S)|)/2 + 2.$$

Thus,

$$|M_3(S)| + |M_4(S)| + |M_5(S)| + |M_6(S)| =$$

$$(3|M_3(S)| + 4|M_4(S)| + 5|M_5(S)| + 6|M_6(S)|)/4 + 2.$$

Then, $|M_3(S)| = 8 + |M_5(S)| + 2|M_6(S)|$. $\diamond$

According to Lemma 5.3.3, we have:

Theorem 5.3.1. *Any simple closed surface at least has eight corner surface points.*

Furthermore,

Theorem 5.3.2. *Any simple closed surface at least has 14 points.*

Proof: First, any two corner surface points cannot be adjacent; otherwise, there would be a three-dimensional unit cell in the surface. According to Theorem 5.1.2, each simple closed surface has at least eight corner points, each of which is connected to three edges, which can not connect with corner surface points. Therefore, there are $24 = (3 \times 8)$ points, each of which is not a corner surface points. So, the number of the points in a closed surface is:

$$|M_3(S)| + |M_4(S)| + |M_5(S)| + |M_6(S)| =$$

$$8 + |M_5(S)| + 2|M_6(S)| + |M_4(S)| + |M_5(S)| + |M_6(S)| =$$

$$8 + |M_4(S)| + 2|M_5(S)| + 3|M_6(S)|$$

and

$$4|M_4(S)| + 5|M_5(S)| + 6|M_6(S)| \geq 24.$$

According to Lemma 5.3.3, we consider the following integer programming problem:

$$8 + |M_4(S)| + 2|M_5(S)| + 3|M_6(S)| = min$$

$$4|M_4(S)| + 5|M_5(S)| + 6|M_6(S)| \geq 24;$$

we have got $|M_4(S)| = 6$ and $|M_5(S)| = |M_6(S)| = 0$. That is, any simple closed surface at least has 14 points. $\diamond$

5.4 General Digital Surfaces in 3D

In this section, we will discuss the general digital surface in general/indirect adjacency. In Chapter 8, we will present that the difference between the purely direct adjacency and indirect adjacency is to use sub graphs vs. partial-graphs to form the basic-elements/cells.

Why do we need more on general digital surfaces? the reason is that Morgenthaler-Rosenfeld's definition that has totally nine types of variations is not complete, see Fig. 4.1.

Kong and Roscoe presented a detailed work to analyze Morgenthaler-Rosenfeld's surfaces [48]. Since Kong and Roscoe have mapped the Morgenthaler-Rosenfeld's simple surface points into continuous space, some concepts and results come to be more intuitive but the method used in [48] turned to be more and more difficult to understand. On the other hand, it is difficult to deal with the algorithms based on such analog results. Chen has continued to the study of all cases of (α,β)-surfaces and to eliminate some (other) trivial and overlapping cases [13].

Section 5.4.2 introduces Kong and Roscoe's analog method. In Section 5.4.2, we will present the real useful types of (α,β)-surfaces. In Section 5.4.3, we will develop or modify three algorithms for the three definitions in [57], [48], and [28], respectively.

Because each type of these (α,β)-surfaces has advantages and disadvantages in real image processing, and some visually true digital surface (or surface points) is not included in any of (α,β)-surfaces (or surface points), for this reason, we suggest a more general definition for digital surface in both direct and indirect adjacency in Section 5.4.4. In order to develop a fast algorithm for real surface tracking, a quasi-digital-surface is defined even though sometimes the definition is too weak in terms of mathematics.

5.4.1 (α,β)-Surfaces and Continuous Analogs

As we did in Chapter 2, let S be a subset of Σ_3. Two distinct points p and q in S are α-connected, α=6, 18, or 26, if there is a sequence of points (called a path) Π, $p = p_0, p_1, ..., p_n = q$, where p_i and p_{i+1} are α-adjacent where $i = 0, ..., n-1$. It is easy to know that α-connectedness is an equivalence relation. So, by using α-connectedness, any subset of Σ_3 can be partitioned into several α-connected subsets (called α-components). We say S is α-adjacent to a point p if a point in S is α- adjacent to p.

Let p be a point (we always assume that p is not at the border of Σ_3), $N_p = N(27, p)$ denotes the set of points containing p and all 26-adjacent points of p. N_p is the neighborhood of p in terms of digital space. If p is in S, define $S(p) = S \cap N_p$. We repeat the famous Morgenthaler-Rosenfeld definition here.

Definition 5.4.1. *(Morgenthaler-Rosenfeld) A point p in S is an (α,β)-simple surface point (or (α,β)-surface point) if and only if*

(1) $S(p)$ is an α-component,

(2) $N_p - S(p)$ has exactly two β-components and each of which is β-adjacent to p,

(3) Each of the α-neighbors (of p) is β-adjacent to both β-components of $N_p - S(p)$.

If every point of S is an (α,β)-surface point, then S is called an (α,β)-surface.

It is easy to know that an (α,β)-surface is a closed surface. This definition was the first formal definition of digital surfaces. The definition provides a direction of theoretical studies on digital surfaces. However, Morgenthaler, Reed and Rosenfeld defined their "simple surface points axiomatically, and it is difficult to understand this concept just by reading the axioms [48]."

Kong and Roscoe embedded the Σ_3 into R^3 to present a visual interpretation of (digital) surface points. Because some concepts defined in [48] are very complicated, we just extract the most useful and meaningful concepts for simplicity. However, the simplification may not be strict in mathematics. One could find a complete definition in [48]. The rest of this section will roughly review the main concepts and results of their paper [48].

In [48], a polyhedral surface is defined as a set of triangles $\{T_i\}$ in R^3 that satisfies

(1) $T_i \cap T_j$ is an empty set, or a side or corner point of both T_i and T_j for $i \neq j$.

(2) Each side of a triangle is a side of at most one (other than this triangle) in the triangle set. A plate π is a polyhedral surface such that the polyhedral surface is in a unit-cube. The boundary of π , $\partial\Pi$, is on the boundary faces of the unit-cube, and each vertex of $\partial\Pi$ is on the corner of the unit-cube.

Let p be a point in Σ_3. A plate cycle at p is a sequence of distinct plate $\{\pi_i | i = 0, ..., n\}$:

(1) There is a sequence $\{e_i | i = 0, ..., n\}$ in which e_i, e_{i+1} are distinct edges of π_i, $i = 0, ..., n-1$, e_n and e_0 are distinct edges of π_n, and p is an endpoint of each e_i.

(2) If $i \neq j$, then $\pi_i \cap \pi_j$ is the union of a number of straight line segments, each of which is an edge of both plates and a set of points each of which is a vertex of both plates.

(3) Any edge of a π_i is an edge of at most one other π_j.

The concept of plate cycles plays a very important rule in [48]. It is also very intuitive to describe what does a point on a digital surface look like. A plate cycle of p is a two-dimensional neighborhood at p. However, this concept has no algorithmic advantages because to decide if a plate set is a plate cycle is time consuming. We will discuss this issue in Section 5.4.3. In [48], Kong and Roscoe obtained a series of results on (α,β)-surfaces:

Proposition 5.4.1. *Case (α,26)) If $p \in S$ and $S \subset \Sigma_3$, then p is an (α,26)-surface point of S iff the following all hold:*

(i) No unit-cube in N_p contains eight points in S.

(ii) $F_{26}(S(p)) < p >$ is the plate set of a single plate cycle at p.

(iii) If q is an α-neighbor of p that is contained in S, then q is a vertex of some plate in $F_{26}(S(p)) < p >$.

In the above proposition, $F_{26}(T)$ is defined as the set of 1×1 squares whose corners all lie in T. If G is a set of plates, $G < p >$ is all plates containing p.

Proposition 5.4.2. *Case (α,18)) If p is an (α,18)-surface point of a set S, and all α-neighbors of p that lie in S are also (α,18)-surface points of S, then the following all hold:*

(i) Each unit-cube in N_p contains at most six points in S.

(ii) $F_{18}(S(p)) < p >$ is the plate set of a single plate cycle at p.

(iii) If q is an α-neighbor of p that is contained in S, then q is a vertex of some plate in $F_{18}(S(p)) < p >$.

Conversely, if $p \in S$, $S \in \Sigma_3$, *and (i),(ii),(iii) all hold, then p is an (α,18)-surface point of S.*

In the above proposition, $F_{18}(T)$ is defined as the set of 1×1 squares or compound plates whose corners all lie in T. A compound plate CP is defined as:

(1) has six vertices in a unit-cube,

(2) two vertices not contained in (1) have distance $\sqrt{3}$,

(3) let g be the centroid of the unit-cube, then,

$$CP = \cup\{triangle(uvg) |\ u,v \text{ are 6-adjacent points in that 6 vertices}\}.$$

See Fig 5.17.

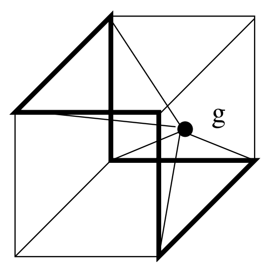


Fig. 5.17. A compound plate in a unit-cube (3-cell).

Obviously, a compound plate has just one possible interpretation for the six vertices set in a unit-cube. However, one can give other reasonable interpretations. Some of the interpretations do not satisfy Proposition 5.4.2.

Proposition 5.4.3. *(Case (18,6)) Suppose that p is an (18,6)-surface point of a set S, and all 18-neighbors of p that lie in S are also (18,6)-surface points of S. Then the following all hold:*

(i) Each unit-cube in N_p *contains at most four points in S and every unit-cube containing four points in S is identical to unit-cube (a), (b), (c), or (d) in Fig. 5.18.*

(ii) No two unit-cubes sharing a face is as (f) in Fig. 5.18.

(iii) $F_6(S(p)) < p >$ *is the plate set of a single plate cycle at p.*

(iv) If q is an 18-neighbor of p that is contained in S, then q is a vertex of some plate in $F_{18}(S(p)) < p >$.

Conversely, if $p \in S$, $S \subset \Sigma_3$, *and (i),(ii),(iii), and (iv) all hold, then p is an (18,6)-surface point of S.*

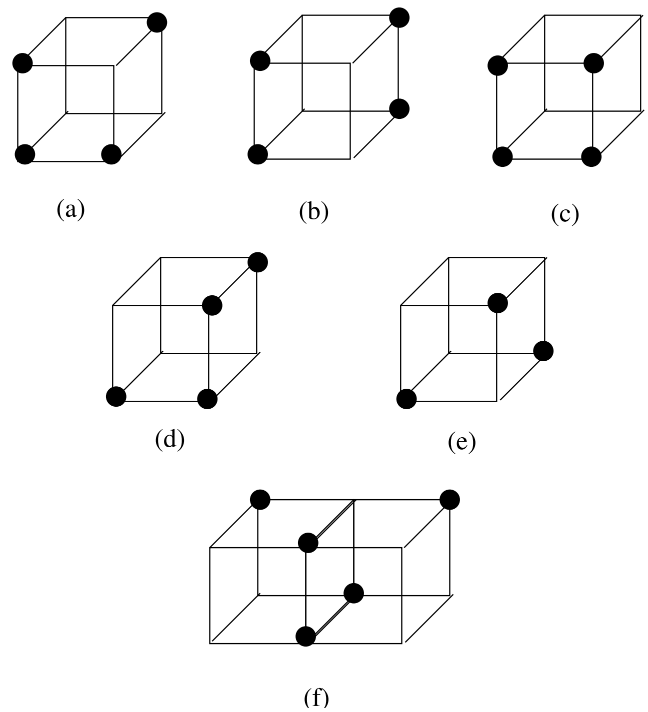


Fig. 5.18. 18-neighbors where points in S are marked as dots.

Proposition 5.4.4. *(Case (26,6)) Suppose p is a (26,6)-surface point of a set S, and all 26-neighbors of p that lie in S are also (26,6)-surface points of S. Then the following all hold:*

(i) Each unit-cube in N_p contains at most four points in S and every unit-cube containing four points in S is identical to unit-cube (b) or (d) in Fig. 5.18.

(ii) $F_6(S(p)) < p >$ is the plate set of a single plate cycle at p.

(iii) If q is a 26-neighbor of p that is contained in S, then q is a vertex of some plate in $F_{18}(S(p)) < p >$.

Conversely, if $p \in S$, $S \subset \Sigma_3$, and (i),(ii),(iii), and (iv) all hold, then p is a (26,6)-surface point of S.

Here, $F_6(T)$ is defined as the collections of the following plates:

(1) Plate π is the union of triangles $\triangle ABC$ and $\triangle BCD$, where A, B, C, and D are the four points in a unit-cube of Fig. 5.18 (a) which are in S, and $BC = \sqrt{3}$.

(2) Plate π is an $1 \times \sqrt{2}$ rectangle whose corners are the four points in a unit-cube of Fig. 5.18 (b) which are in S.

(3) Plate π is a $(\sqrt{2}, \sqrt{2}, \sqrt{2})$-triangle whose corners are the three points in a unit-cube of Fig. 5.18 (e) which are in S.

(4) Plate π is an 1×1 rectangle whose corners are the four points in a unit-cube of Fig. 5.18 (c) which are in S.

A (6,6)-surface does not have the same results as shown above because there is no (6,6)-surface with finite points [48]. In addition, each (18,26) or (26,26)-surface must be a coordinate plane [48]. Each (26,18)-surface is a coordinate plane or contains only compound plates. In the next section, we will discuss other trivial cases to make (the remaining) (α,β)-surfaces to be really meaningful.

5.4.2 Nontrivial Cases of (α,β)-Surfaces

For the nine different categories of (α,β)-surfaces, α, β=6,18,26. Kong and Roscoe [48] already discovered some of the cases are not real exist or in some trivial forms. These cases are not really useful for image processing and computer graphics. These cases include (6,6)-surfaces, (18,26)-surfaces, and (26,26)-surfaces [48]. In addition, (26,18)-surfaces are either coordination plane or only consist of compound plates [48]. Therefore, (α,26)-surfaces can be reduced to just (6,26)-surfaces [13].

(6,18)-surfaces. This section will find more trivial (α,β)-surfaces. First, we consider (6,18)-surfaces and (18,18)-surfaces. Obviously, a (6,18)-surface is a (6,26)-surface, and it cannot contain any corner point as follows:

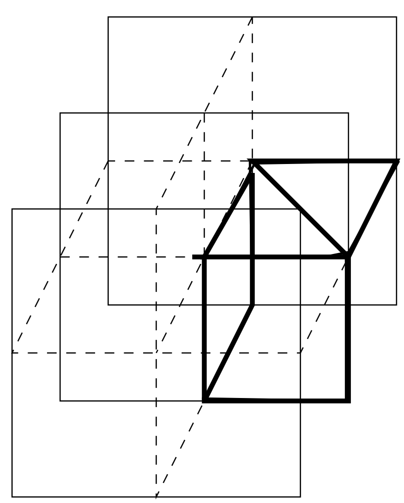


Fig. 5.19. A corner surface point structure.

Thus, a (6,18)-surface is a coordination plane or some simple "stage surfaces" just as shown in Fig. 5.20.

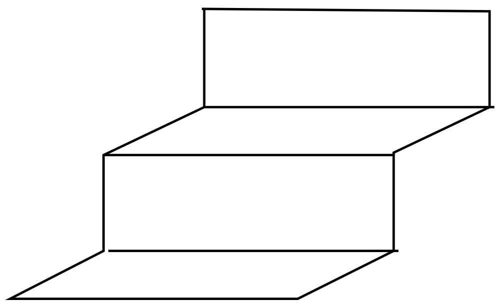


Fig. 5.20. The general structure of (6,18)-surfaces.

(18,18)-surfaces. For (18,18)-surfaces, we can have:

Proposition 5.4.5. *If a compound plate PC shares an edge with an 1×1 plate PW, then there must be a point in $PC \cup PW$ that cannot be a surface point.*

Proof: Assume a compound plate in S is as shown in Fig. 5.21. Each edge has only two choices to be shared with an 1×1 plate. Therefore, suppose (p, a) is the edge. If $\{p, q, b, a\}$ is in S, then $\{c, d, p, q\}$ must be in S according to Proposition 5.4.2. As we know, b is also a surface point, so $\{c, f, b, q\}$ is in S. Thus, p cannot be an (18,18)-surface point. We can get the same result for other cases. ⋄

According to the above proposition, we have

Proposition 5.4.6. *(18,18)-surfaces either only consist of 1×1 plates or only consist of compound plates.*

One could easily prove that:

Proposition 5.4.7. *If an (18,18)-surface only consists of 1×1 plates, then it is a coordinate plane.*

Proposition 5.4.8. *There is only one closed (18,18)-surface based on the definition of compound plates (shown in Fig. 5.22).*

Proof: There are two methods to prove this proposition. For the first method, one can start with a compound plate and then try to link another compound plate, and so on. We can see there is only one solution like Fig. 5.22. The second method is to use the Euler planar graph theorem, i.e., "$|E| + 2 = |F| + |V|$." where E is the edge set, F is the face set, and V is the vertex set.

Let's consider the second method because it is more elegant.

Suppose that S is a closed (18,18)-surface by compound plates. S must be a planar graph. Thus,

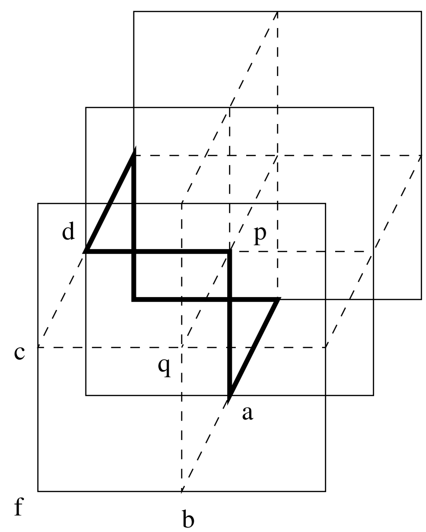


Fig. 5.21. A compound plate in N_p.

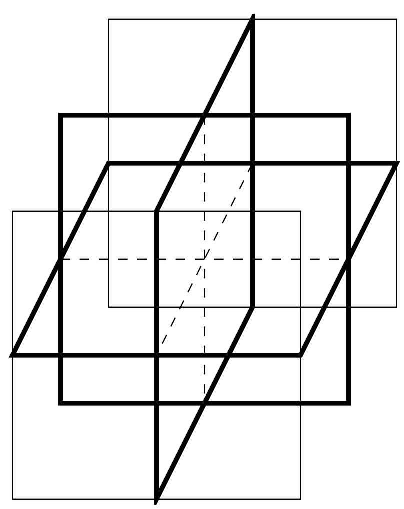


Fig. 5.22. A closed (18,18)-surface.

$$|E| + 2 = |F| + |V|.$$

On the other hand, first, a face has six edges and an edge is included in exactly two faces, so

$$6|F| = 2|E|.$$

Second, there are only two types of vertices. First, a vertex is in two faces. Second, a vertex is in four faces. Assume the set of the first type vertices is V_1 and the set of the second type vertices is V_2. Then,

$$|V_1| + |V_2| = |V|.$$

We can see, if two compound plates share an edge, then they share two edges. In addition, these two edges share a point which is only contained by two faces. So, a compound plate must share with three other compound plates. Therefore, in a face, there are three points in V_1 and three points in V_2. So,

$$3|F| = 2|V_1| \text{ and } 3|F| = 4|V_2|.$$

We have $|V_1| = 2|V_2|$. We can get $|F| = 8$ and $|E| = 24$ and $|V| = 18$. That is the case of Fig. 5.22. $\diamond$

Similarly, we can also prove that "there is only one type of closed (26,18)-surface by compound plates." Therefore, (α,18)-surfaces are not really useful in image processing.

(18,6)- and (26,6)-surfaces. A (26,6)-surface is an (18,6)-surface [48]. Since (α,18)-surfaces are useless surfaces, (α,26)-surfaces can be reduced to (6,26)-surfaces, and (α,6)-surfaces can be reduced to (18,6)-surfaces.

Therefore, there are only two useful types of (α,β)-surfaces: (6,26)- and (18,6)-surfaces.

5.4.3 Algorithmic Considerations and Surface-Cell Traversal Algorithms

As we discussed above, Kong-Roscoe's analog results are very helpful for understanding the Morgenthaler- Rosenfeld's surfaces. An algorithm to find a plate cycle was discussed in [48]. However, if one performs this algorithm on each point of set S, it will be very slow because for each point one needs to find a plate cycle. We will show in this section that this algorithm is not faster than testing all conditions of Morgenthaler-Rosenfeld's surfaces as discussed by Reed and Rosenfeld [63]. On the other hand, using continuous spaces to analog can help people to understand and reduce some ambiguities. However, discrete spaces and objects have their own properties. The continuous analog is just a single interpretation of a digital object, and it may have many interpretations.

Improvement of the Algorithm by Morgenthaler-Reed-Rosenfeld. The idea of the algorithm by Morgenthaler-Reed-Rosenfeld is to test all three conditions for each point in S in Definition 4.1.2. The main task of the algorithm is to determine all "1" points is an α-component and all "0" points in N_p form two disconnected α-components. By using Tarjan's depth-first-search or breadth-first-search technique, the theoretically fastest algorithm to search connected components can be developed [31].

Even though one could use Tarjan's technique, this algorithm is still not very fast practically. The reason is that for each point one needs to find three components in N_p . Many steps are repeated because of adjacency in S. For example, assume p has four α-adjacent points q_1, q_2, q_3, q_4 in S, then the algorithm must reconsider p at least four times when the algorithm goes through q_1, q_2, q_3, q_4.

The following algorithm, Algorithm 5.4.1, uses C_1, C_2 to store two "0" value components (complement components in N_p) and A to store "1" value component. In order to reduce the repeated actions, after we finish a test (for three conditions in definition 5.4.1) for point p, we move to an α-adjacent point in S of p. Meanwhile, we do not empty C_1, C_2 and A, but update them.

Algorithm 5.4.1

Step 1. Assign a point of S to p , and let C_1, C_2 , and A be empty sets. $STACK$ is a stack. Get C_1, C_2 and A for p.
Step 2. Examine all three conditions in definition 5.4.1. If p is not a surface point, halt.
Step 3. Mark p, and push all unmarked α-adjacent points into the stack $STACK$.
Step 4. Pop up a point p' from $STACK$ (this point is an α-adjacent point most of the time). If p' is adjacent to p, $N_p \cap N_{p'}$ has 18, 12, or 8 points depending on their adjacency. Get C_1, C_2 , and A for p'. Then $p \leftarrow p'$.
Step 5. Goto Step 2 until $STACK$ is empty.

Modifying the Algorithm made by Kong-Roscoe. Kong and Roscoe presented an algorithm to find a plate-cycle for a point p in [48]. p is a surface-point if $P < p >$, all plates containing p is a plate-cycle. Note, $P < p >$ can also be defined as 1×1 square for (6,26)-surface, and $P < p >$ can also be defined as one of the cases in Fig. 5.20 (a)—(e) for (18,6)-surfaces. The aim of the original algorithm by Kong-Roscoe is to find a largest plate cycle for a point p. We modify the algorithm and give the result as follows [48]:

Algorithm 5.4.2

Step 1. Let π' and π'' be in $P < p >$ and π' and π'' shares an edge containing p . Let $\pi_0 = \pi'$ and $\pi_1 = \pi''$. Let e_0 be the common edge containing p.
Step 2. For $i = 1, 2, 3,$ if there exists an e_i and it is an edge of π_i that contains p but is different from e_{i-1}, and there exists a plate π_{i+1} (which is different from π_i) such that e_i is an edge of π_{i+1}. Until there is $m < i$ such that $e_i = e_m$.
Step 3. If all plates used in Step 2 are equal to the set of $P < p >$, then p is a surface point. (One just needs to test if $|P < p >|$ is equal to m.)

This algorithm needs to get $P < p >$ for each p and store the set first. In Step 2, one needs to select an appropriate π_i . For (6,26)-surfaces, we can determine all 1×1 squares containing p to be $P < p >$ in N_p. For (18,6)-surfaces, it is not easy to determine $P < p >$ because there are many different kinds of plates. We can also see that this algorithm (in both cases) is not a fast algorithm.

Both Algorithms 5.4.1 and 5.4.2 are not very fast because their definitions are based on local properties, i.e., they use surface-point to define a surface. In such a case, a local neighborhood N_p and $S \cap N_p$ must be considered extensively. In addition, such a definition cannot deal with boundary, so each surface under this definition must be a closed surface.

Chen and Zhang showed a definition of digital surfaces that surmounts these two problems in direct adjacency in Chapter 4. We will discuss the indirect adjacency later in this Chapter.

Surface-Cell Traversal Algorithm for (6,26)-surfaces. We have proved that a (6,26)-surface is the regular parallel-move based surface in Section 5.3. According to the proof of Theorem 5.3.1, we can easily get:

Lemma 5.4.1. *Let S be a digital surface by parallel-move. If S does not contain any point p such that N_p is equivalent to Fig. 5.23, then every point in S is regular.*

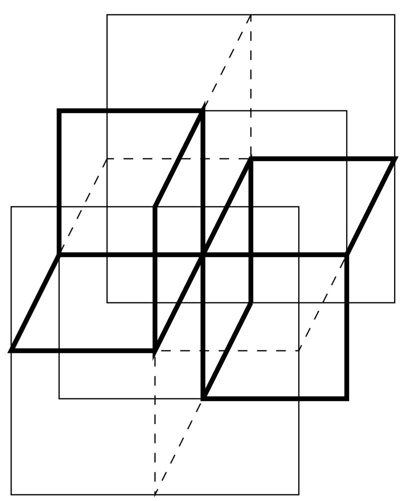


Fig. 5.23. A non-regular inner surface point.

An algorithm based on Tarjan's breadth-first-searching technique can be developed to decide or track S in a global way, which does not examine each

N_p for S. An algorithm that traverses every surface-cell using a parallel-move of line-cell was described in Section 4.1. We add the last step to decide if S contains the sitution in Fig. 5.23.

Algorithm 5.4.3: Surface-Cell Traversal Algorithm.

Step 1. Start with two 6-adjacent points $LC = \{p, q\}$ (a line-cell) in S.
Step 2. Put all (at most two if S is a surface, otherwise halt) parallel-moves of $\{p, q\}$ in a stack $STACK$.
Step 3. Mark line-cell $LC = \{p, q\}$. Pop up a line-cell to LC from $STACK$, repeating Step 2.
Step 4. S is a surface if all line-cells in S have been marked.
Step 5. Check all unit-cubes, each of which contains a point in S. If a cube contains seven points in S, assign p as the point whose diagonal point in the cube is not in S. If p has any 6-adjacent points not in the cube, then S is not a Morgenthaler-Rosenfeld's surface.

This algorithm is very simple and is optimized. However, there are two problems: (1) This algorithm cannot be used for (18,6)-surfaces; and (2) the surfaces defined by Definition 4.1.3 varied a little from Morgenthaler-Rosenfeld's surfaces.

Again, we can see the Surface-Cell Traversal Algorithm is much simpler and faster than the other two. However, the disadvantage is that the Surface-Cell Traversal Algorithm is only for (6,26)-surfaces, ((6,26)- surfaces are the most important digital surfaces.) In the following section, we will discuss a more general definition of digital surfaces.

5.4.4 A New Definition of General Digital Surfaces

Since (α,18)-surfaces are useless surfaces, (α,26)-surfaces can be reduced to (6,26)-surfaces, and (α,6)- surfaces can be reduced to (18,6)-surfaces. Therefore, there are only two useful types of (α,β)-surfaces: (6,26)- and (18,6)-surfaces.

In addition, there are many visually true surfaces (or surface points) that cannot be contained by any type of (α,β)-surfaces. We have showed such an example, Fig. 4.1, in Chapter 4. Let us see another example, Fig. 5.24 shows such a case $S(p)$ in N_p. We can see $S(p)$ is 18-connected (it is also 6-connected and 26-connected). β can not be 18 or 26; otherwise, there are no two β- components in $N_p - S(p)$. The only choice is $\beta = 6$. If $\beta = 6$, q can not be β(=6)-adjacent to both β-components C_1 and C_2 in $N_p - S(p)$. So it is not any (α,β)-surface point.

Therefore, we can see that the (α,β)-surface is not a complete description of digital surfaces. An (α,β)- surface has blind spots and weakness in representing some digital surfaces, especially in computer graphics.

On the other hand, indirect adjacency causes ambiguous interpretations for a set S. Therefore, 18- or 26- adjacency introduces ambiguities for the

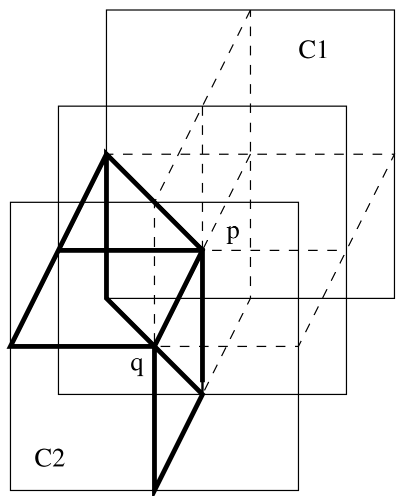


Fig. 5.24. A visually true surfaces point, but not an (α,β)-surface point for any α and β.

interpretation of surfaces. For example, suppose that we allow 18-adjacency for a set S containing points as shown in the following figure, Fig. 5.25 (a). From it, we can see that for the 6-adjacency we have a unique interpretation. However, for 18-adjacency, we could have an interpretation like (b), which is a surface. If we interpret (a) like (c), it is not a surface. If we interpret (a) to be (d), it is a different surface.

All of these interpretations are reasonable under 18-adjacency. That is why a general solution allowing indirect adjacency is very difficult. As we know, there are only six types of simple surface points for direct adjacency [29][51], but it is really hard to get all simple surface points allowing indirect adjacency. The ambiguity is real, and it is impossible to reduce these cases to be unique. On the contrary, we could see the power of the indirect adjacency.

This section will give a new definition of general digital surfaces for all 6-, 18-, and 26-connectivity. Based on the previous work of [57], [48], and [28], the (α,β)-surfaces are considered to be special cases in the new definition.

Definition 5.4.2. *An α-surface-cell is a polygon on a plane in a unit-cube. Each pair of adjacent points on the polygon is α-adjacent. Such a pair is called an edge or line-cell of the α-surface-cell.*

In addition, we could define an α-surface-cell to be an minimized closed α-adjacent path, i.e., there is not pure subset of the path such that this subset can be a closed α-adjacent path.

Definition 5.4.3. *A set S is a digital surface (α-surface) if there is a collection of α-surface-cells in S, denoted by $U(S)$ and satisfying:*

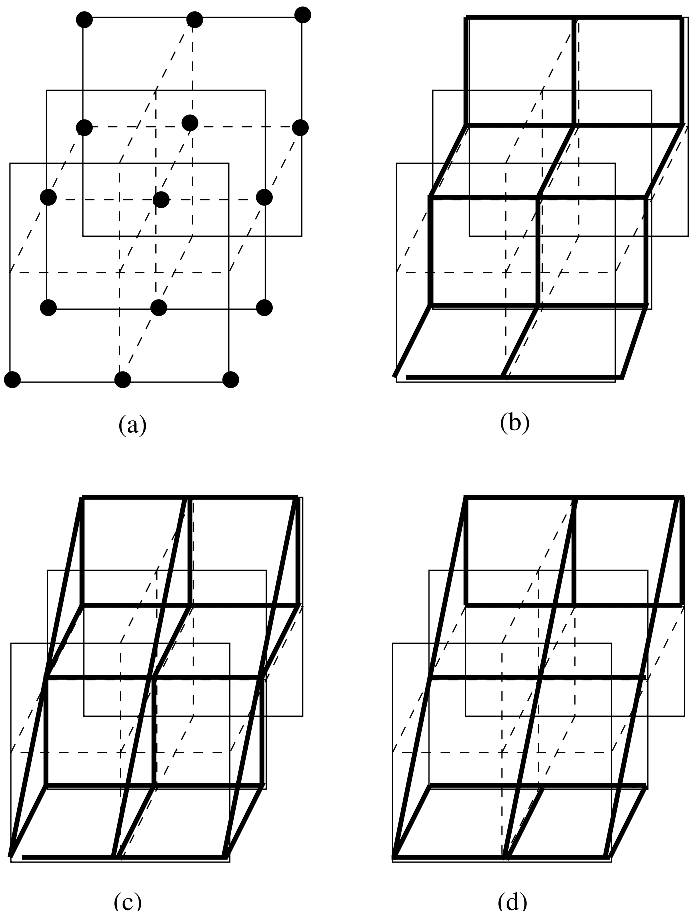


Fig. 5.25. A digital points set in S and some of the interpretations.

(1) $S = \cup\{p|\ p$ is a point of an α-surface-cells in $U(S)\}$,
(2) $U(S)$ is line-connected in S.
(3) Each line-cell or edge of a surface-cell is contained by one or two α-surface-cells in $U(S)$.
(4) S does not contain any unit-cube. (It is the only three-dimensional cell we defined in this chapter.)

Definition 5.4.4. *A point p in an α-digital surface $(S, U(S))$ is a regular-surface point if the set of α-surface-cells containing p is line-connected.*

We can see a $U(S)$ is an interpretation of S. Using this definition, we still cannot get a fast algorithm easily. It is even worse in the design of algorithms. However, it provides a broad new area to study with. The following problems are meaningful.

(1) How many interpretations ($U(S)$'s) for a given S? What is the upper limit?

(2) How to find an optimal solution in some sense (e.g., the one with the minimum area)?

(3) Is the problem of finding all $U(S)$'s an NP-hard problem [1] ?

Even though we need to do more study to obtain a fast tracking algorithm, the following definition is still good for representing surfaces such as for computer graphics. In image processing, we sometimes do not desire a strict digital surface, but we would rather having a fast way to extract a "rough" surface. We would have a simple definition satisfying the human's intuition for surfaces. So, we use a different term here: quasi-surfaces.

Definition 5.4.5. *A point p in S is an α-quasi-surface point if*

(1) $N_p \cap S$ is an α-connected component.

(2) $N_p - S$ has two 6-connected components. (That is to say these two components are not 6- connected.) (3) $N_p \cap S$ does not contain any 3-cell.

Definition 5.4.6. *S is an α-quasi-surface if every point of S is an α-quasi-surface point.*

It is not difficult to see,

Lemma 5.4.2. *An (α,β)-surface is an α-surface by Definition 5.4.2. Any point of an α-surface is an α- quasi-surface point by definition 5.4.4.*